Computing ω-Limit Sets in Linear Dynamical Systems

Emmanuel Hainry

LORIA & Université Henri Poincaré, Nancy, France

Unconventional Computation 2008
Wien, Österreich

What is a dynamical system?

Dynamical systems are

- a configuration space
- a dynamics function: maps a configuration to the following

Interests:

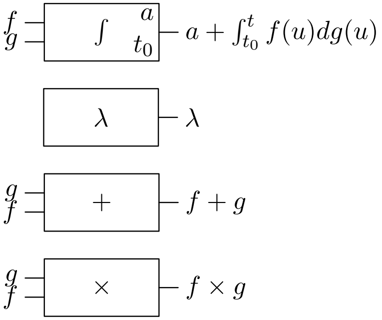
- Describe physical phenomena
- Describe biological phenomena
- Simulate computation models

Natural questions:

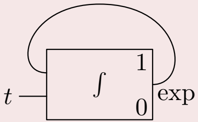
- Population extinction $\longrightarrow$ the system reaches 0.
- A cloud goes over a region $\longrightarrow$ the trajectory intersects a region.
- A program loops infinitely $\longrightarrow$ the system is ultimately periodic.

General Purpose Analog Computer

GPAC [Shannon 41] consists in circuits interconnecting the following components:

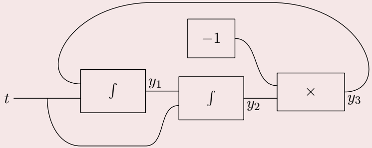


Computing exp with a GPAC



$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

Computing cos with a GPAC



Features of the GPAC

Theorem [Graça Costa 03]

A scalar function $f : \mathbb{R} \to \mathbb{R}$ is generated by a GPAC iff it is a component of the solution of a system

$$y' = p(t, y), \tag{1}$$

where p is a vector of polynomials.

- GPAC is a polynomial dynamical system.

n-body problem

Proposition

The n-body problem (with Newton's laws) can be written as a polynomial dynamical system (with n^2 components)

Theorem [Warren D. Smith]

The n-body problem can "solve" the halting problem for Turing machines in constant time.

- This is a polynomial dynamical system.
- The collapsing of the n-body problem is undecidable.

Problem: Reachability

Definition

Given a dynamical system (X, f), and two points A and B, does the trajectory issued from A reach B?

- Reachability of a point

Problem: ω-limit set

Definition

Given a dynamical system with solution y, the ω-limit set is the set of $A \in X$ such there $(t_n) \to +\infty$ such that $\lim y(t_n) = A$.

- Périodicity, divergence.

Reachability of a hyperplane

Definition

Given a dynamical system (X, f), a point A and a hyperplane $\mathscr{P}$, does the trajectory issued from A intersect $\mathscr{P}$?

- Reachability of a region.
- Skolem-Pisot's problem is equivalent to reachability of a hyperplane for a linear system.

Known result: Undecidability

Theorem

Reachability is undecidable

The halting problem can be written as a reachability question (for a discrete-time or a continuous-time DS).

Known result: Undecidability in Polynomial DS

Proposition

Reachability is undecidable in Continuous-time Polynomial Dynamical Systems.

Proposition

Hyperplane reachability is undecidable in Continuous-time Polynomial Dynamical Systems.

Proofs: From [Bournez, Campagnolo, Graça, Hainry 2007], GPACs and recursive analysis have the same computational power.

Known result: Uncomputability

Proposition

For continuous-time (Polynomial) Dynamical Systems ω-limit sets are not computable.

Recap.

In continuous-time dynamical systems,

	ω-limit set	Reachability	hyperplane reach.
DS	non computable	undecidable	undecidable
polynomial DS	non computable	undecidable	undecidable
linear DS	computable	decidable	?

Linear Dynamical Systems

Definition

A continuous-time linear dynamical system is described by a dimension n, a sqaure matrix A of size n^2 with rationnal coefficients.

$$X = \mathbb{R}^n$$
$$f : Y \mapsto AY$$

A trajectory issued from $Y_0 \in \mathbb{Q}^n$ is a solution of the Cauchy problem: $\begin{cases} Y' = AY \\ Y(0) = Y_0 \end{cases}$. *Id est* $Y(t) = \exp(tA)Y_0$.

Example: prey/predator

Simplified Lottka-Volterra equations:

$$\begin{pmatrix} X \\ Y \end{pmatrix}' = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

With

X	number of predators
Y	number of preys
a	birth rate
b	predation rate

Example: Gravity and wind

$$
\begin{aligned}
y' &= -g \cdot y + v_y \\
x' &= v_x \\
x(0) &= y(0) = 0
\end{aligned}
$$

which can be translated in

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 0 & v_x \\ 0 & -g & v_y \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
$$

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

Reachability

Theorem

Reachability is decidable in continuous-time linear dynamical systems.

$$f : \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ X & \longmapsto & A \cdot X \text{ with } A \in \mathbb{Q}^{n \times n} \end{array}$$

X_0 initial point

Y target

Omega-limit set

Theorem

The ω-limit set is computable for continuous-time linear dynamical systems.

Theorem

The ω-limit set for a continuous-time linear dynamical system is semi-algebraical.

$$f : \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ X & \longmapsto & A \cdot X \text{ with } A \in \mathbb{Q}^{n \times n} \end{array}$$

X_0: initial point

Ω: ω-limit set

Prerequisite

Theorem [Baker]

Given $\alpha \in \mathbb{C}^\star$, either α or $\exp(\alpha)$ is transcendantal.

Theorem [Gelfond-Schneider]

Let α and β algebraic, if $\alpha \notin \{0, 1\}$ and $\beta \notin \mathbb{Q}$, then α^β is transcendantal.

Definition

An algebraic number x is represented by its minimal polynomial χ, a and ϵ such that x is the only root of χ in $\mathcal{B}(a, \epsilon)$

Proposition

$+$, $-$, $\times$, $/$ are computable for algebraic numbers.
Deciding whether an algebraic number is rational is decidable.

Proof (1)

Let us assume the matrix A is in Jordan form:

$$A = \begin{pmatrix} D_1 & 0 & & 0 \\ 0 & D_2 & 0 & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & D_k \end{pmatrix}$$

$$D_i = \begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix}$$

or $D_i = \begin{pmatrix} B & & & \\ I_2 & B & & \\ & \ddots & \ddots & \\ & & I_2 & B \end{pmatrix}$ avec $B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ et $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Proof (trajectory)

$$X(t) = X_0 \exp(tA) = X_0 \begin{pmatrix} \exp(tD_1) & & & \\ & \exp(tD_2) & & \\ & & \ddots & \\ & & & \exp(tD_k) \end{pmatrix}$$

$$\exp(tD_i) = \begin{cases} e^{t\lambda} \begin{pmatrix} 1 & & & & \\ t & 1 & & & \\ \frac{t^2}{2} & t & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \frac{t^k}{k!} & \cdots & \frac{t^2}{2} & t & 1 \end{pmatrix} & \text{or} \\ e^{ta} \begin{pmatrix} B_2 & & & & \\ tB_2 & B_2 & & & \\ \frac{t^2}{2}B_2^2 & tB_2 & B_2 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \frac{t^k}{k!}B_2^k & \cdots & \frac{t^2}{2}B_2^2 & tB_2 & B_2 \end{pmatrix} & \text{with } B_2 = \begin{bmatrix} \cos(tb) & -\sin(tb) \\ \sin(tb) & \cos(tb) \end{bmatrix} \end{cases}$$

Proof (cases)

We can distinguish some cases

- an eigenvalue has a positive real part,
  $\Rightarrow \Omega = \emptyset$
- an eigenvalue has a null real part and > 1 multiplicity,
  $\Rightarrow \Omega = \emptyset$
- all eigenvalues have negative real part,
  $\Rightarrow \Omega = \{0^k\}$
- otherwise...

Proof (otherwise)

All eigenvalues have ≤ 0 real part ; those with null real part have multiplicity 1.

- blocks for eigenvalues with negative real part will converge towards 0.
- blocks for null eigenvalue will be constant.
- other blocks have eigenvalue $\mathrm{i}b$. Those are rotations

If there is only one $\mathrm{i}b$ eigenvalue, the ω-limit set will be a circle. If there are several $\mathrm{i}b_j$, we have to take the commensurability of the b_j into account

Proof (end)

We have several $\mathrm{i}b_j$,
If the b_j are not commensurable (there are no $\alpha_j \in \mathbb{Q}$ s.t. $\sum \alpha_j b_j = 0$),

$$\Omega = \{(x_1, ..., x_n); \forall i, x_{2i+1}^2 + x_{2i+2}^2 = x_{0_2 i+1}^2 + x_{0_2 i+2}^2\}.$$

Otherwise, the commensurability eqs lead to new constraints:

$$\left(\prod_{i<n} X_{0_i}^{\alpha_i}\right) X_{2n}(t)^{\alpha_i} = X_{0_2 n}^{\alpha_n} \prod_{i<n} X_i(t)^{\alpha_i}.$$

where $X_i = x_{2i-1} + \mathrm{i}x_{2i}$

Conclusion

As for discrete DS, reachability is decidable for continuous-time linear Dynamical Systems but undecidable for polynomial DS.

	ω-limit set	Reachability	hyperplane reach.
DS	non computable	undecidable	undecidable
polynomial DS	non computable	undecidable	undecidable
deg.2 poly DS	non computable	undecidable	undecidable
linear DS	computable	decidable	?

Lottka-Volterra equations

$$\begin{pmatrix} X \\ Y \end{pmatrix}' = \begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \underbrace{XY \begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\text{polynomial term}}$$