Self-Assembly of Decidable Sets

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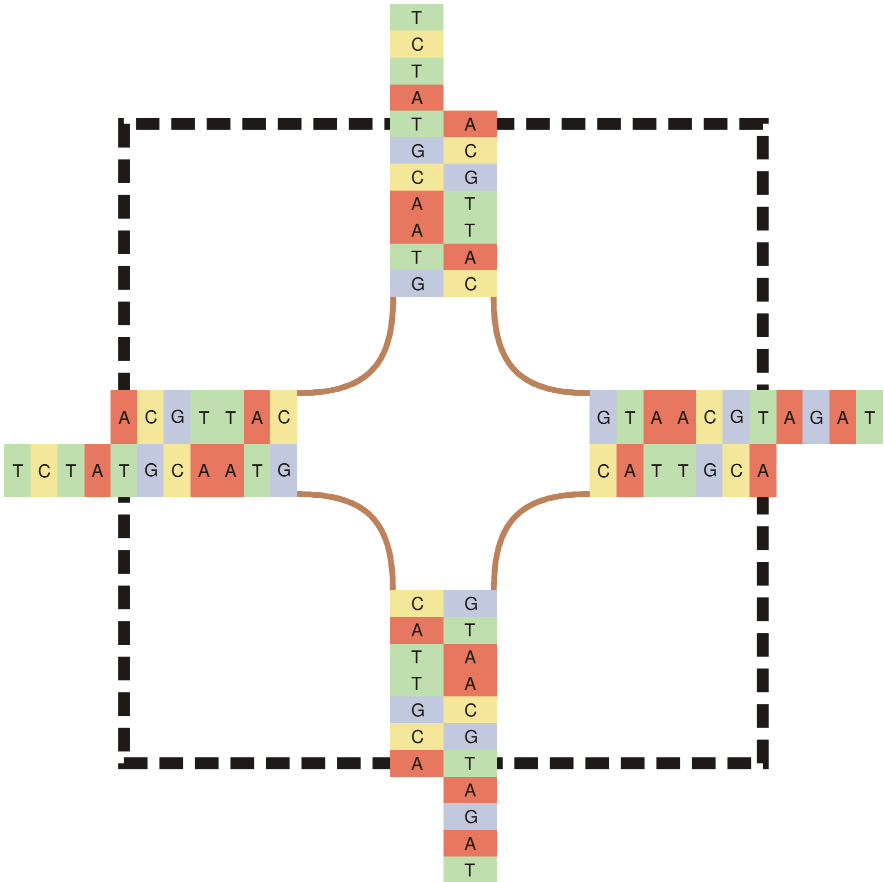
Iowa State University, USA

Outline of Presentation

- Overview of Tile Assembly Model
- The wedge construction
- A new characterization of decidable languages
- Analysis of space requirements

DNA Tile Self-Assembly
Seeman, starting in 1980s

DNA tile, oversimplified:



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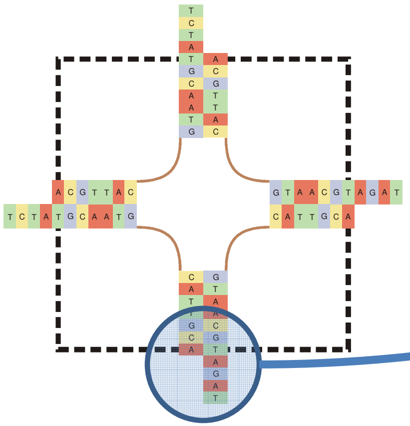

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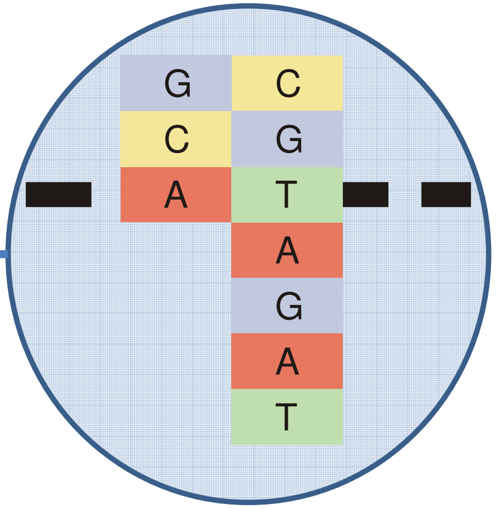

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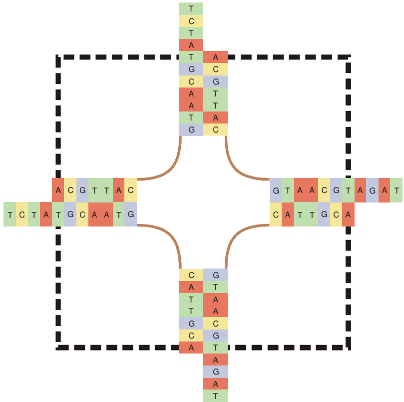

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Choice of sticky ends allows one to *program* the pattern of the array.

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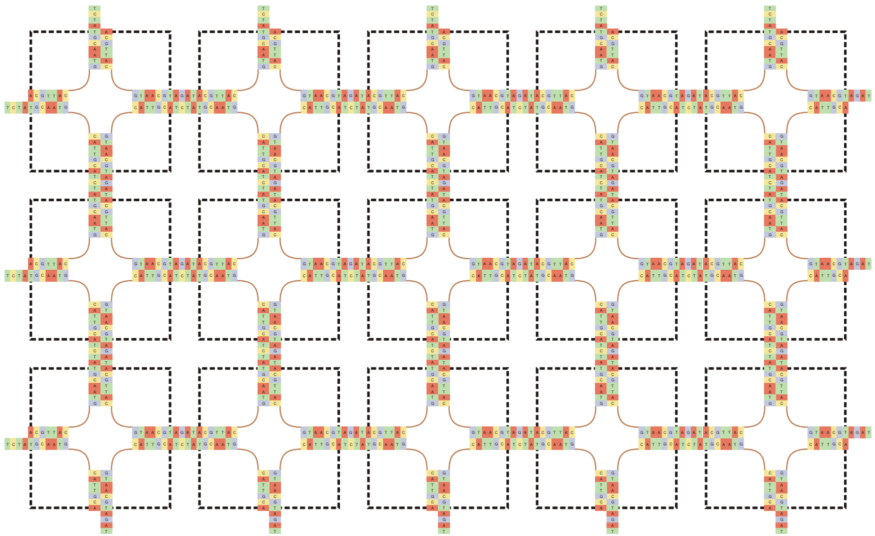

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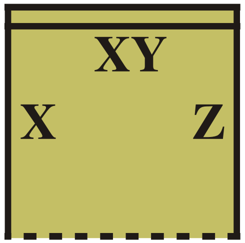

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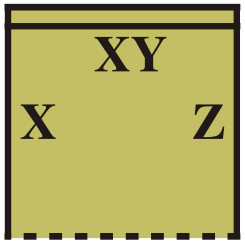

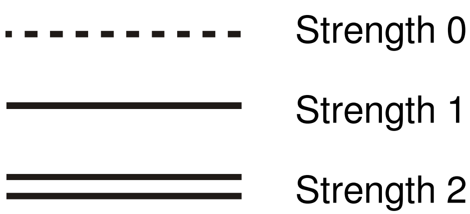

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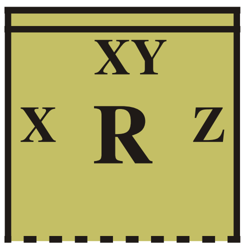

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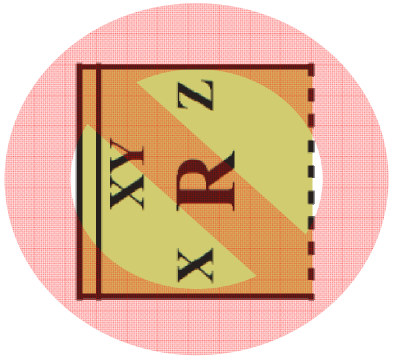

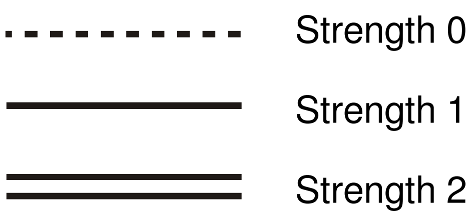

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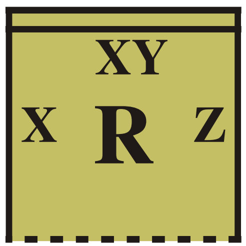

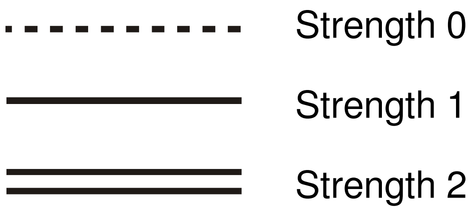

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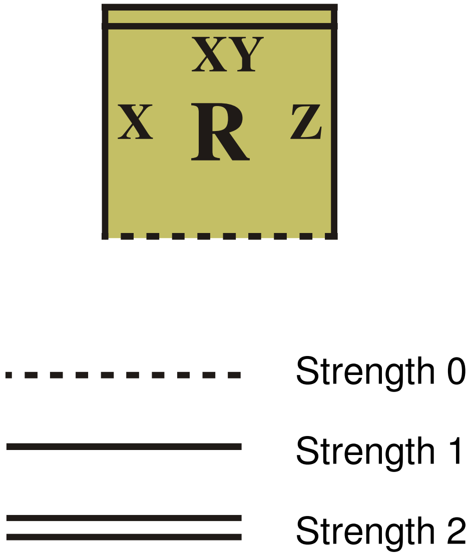

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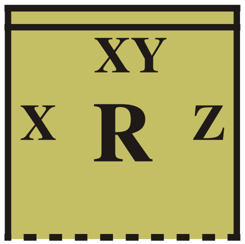

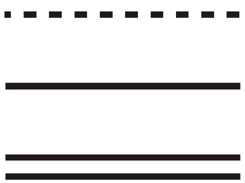

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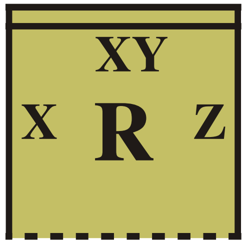

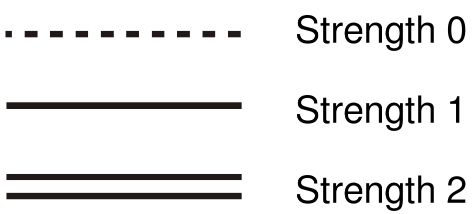

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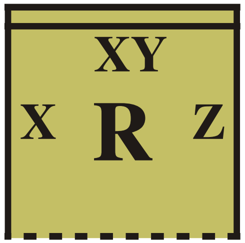

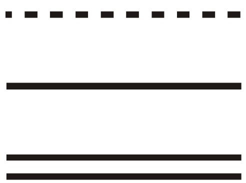

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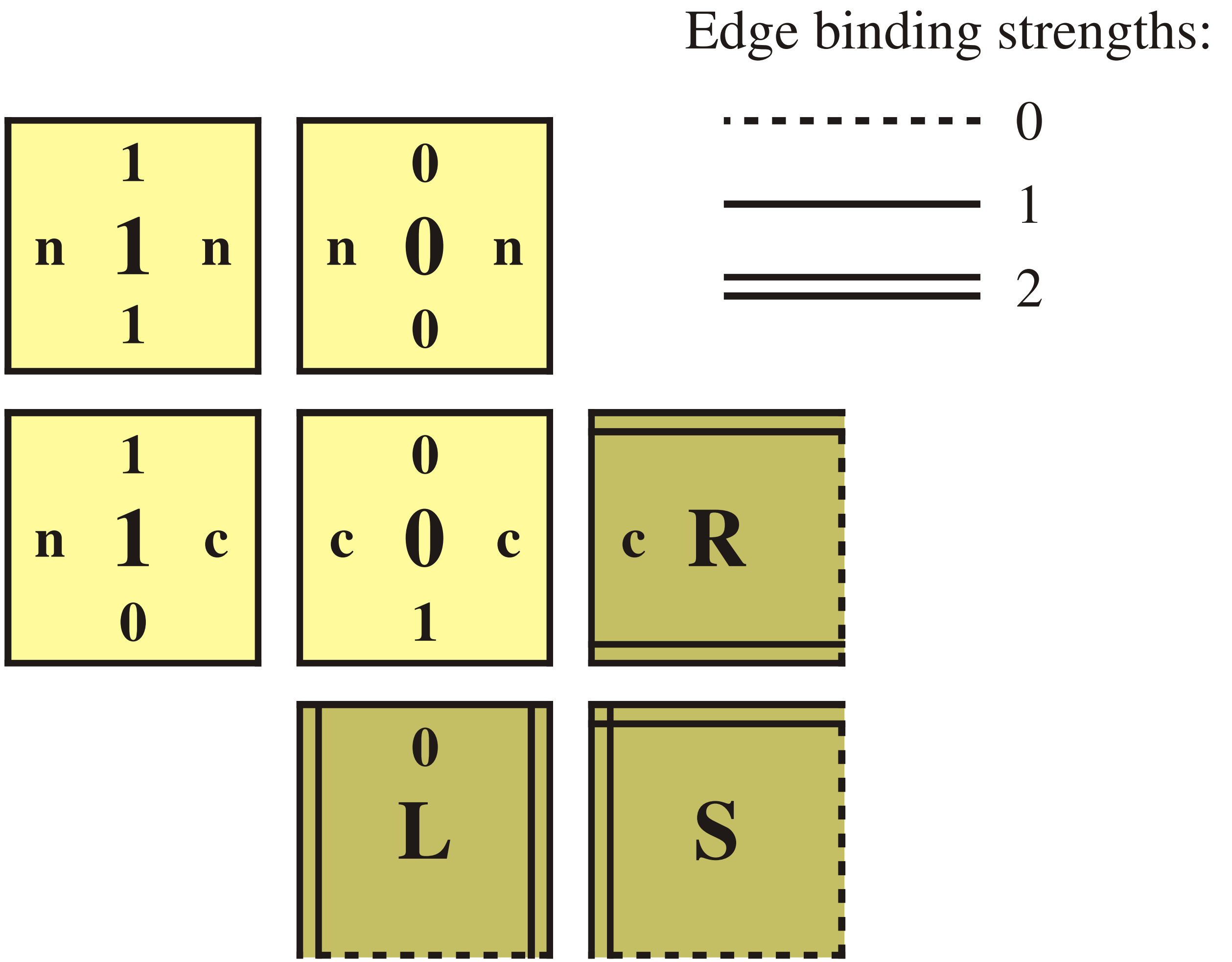
NEXT: An example…

Tile Assembly Example



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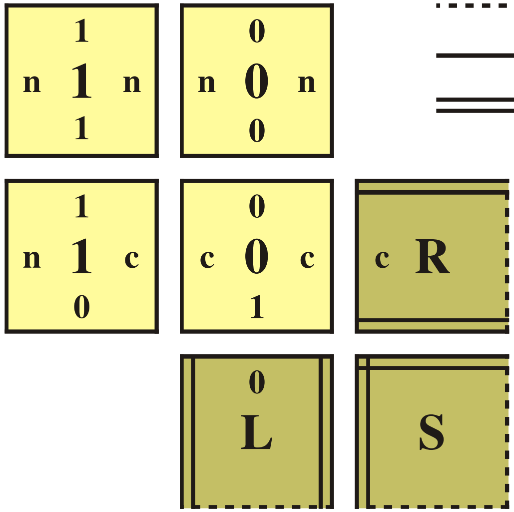
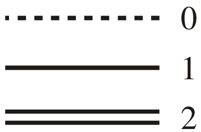
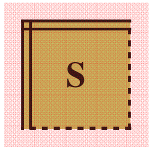


Edge binding strengths:





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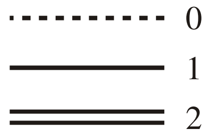

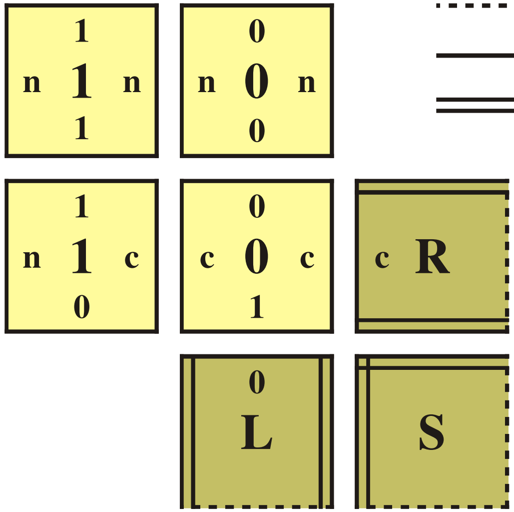

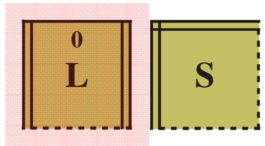

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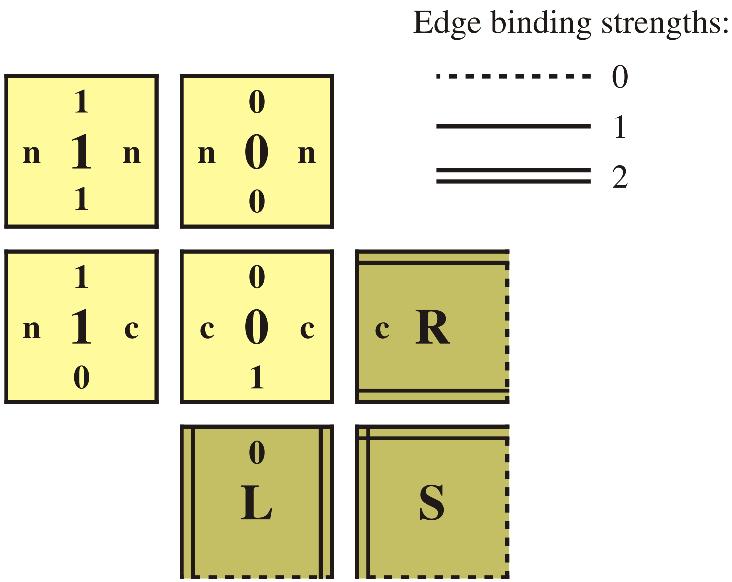

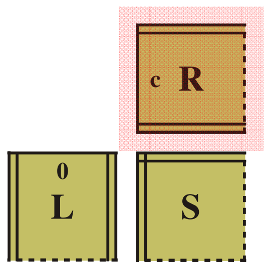

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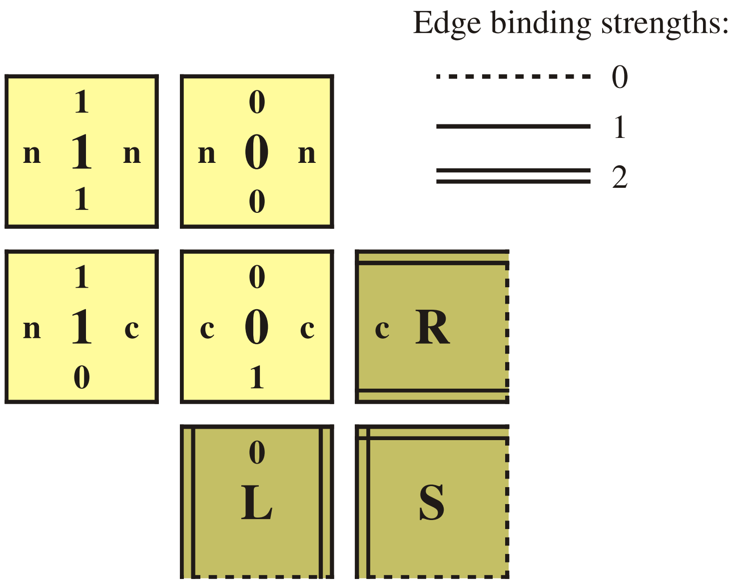

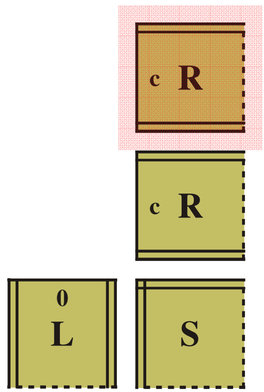

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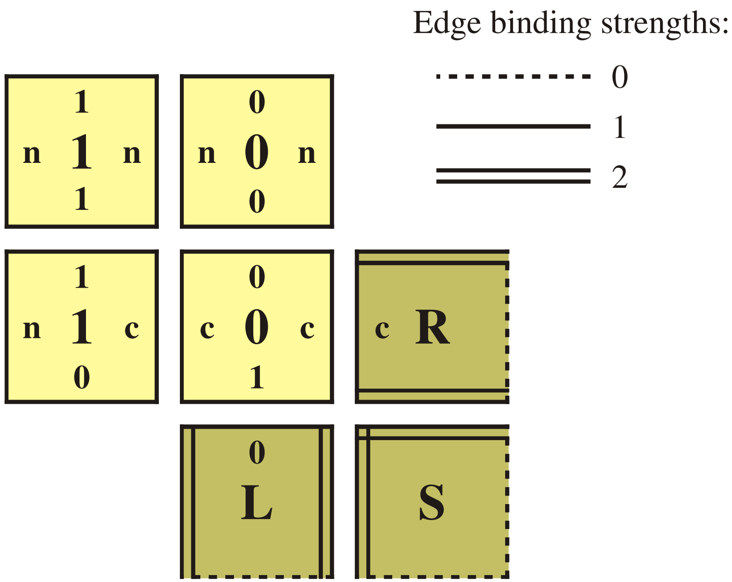

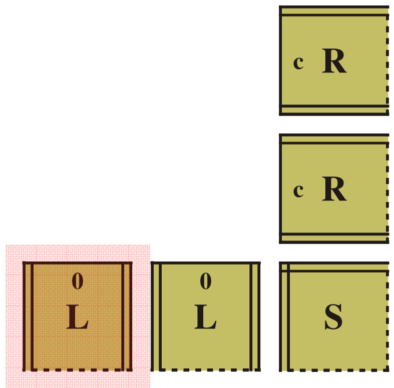

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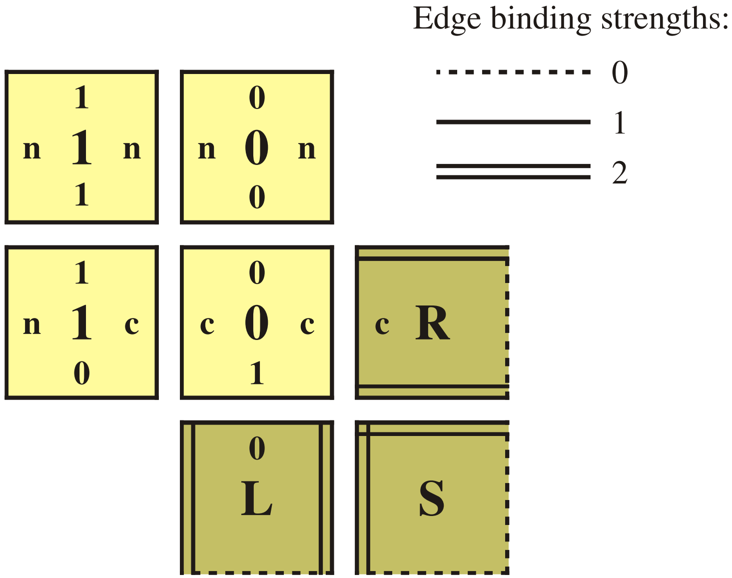

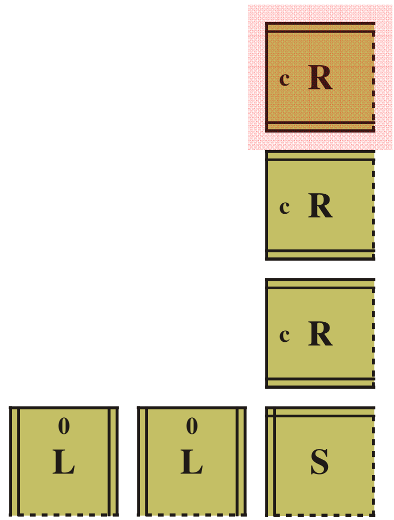

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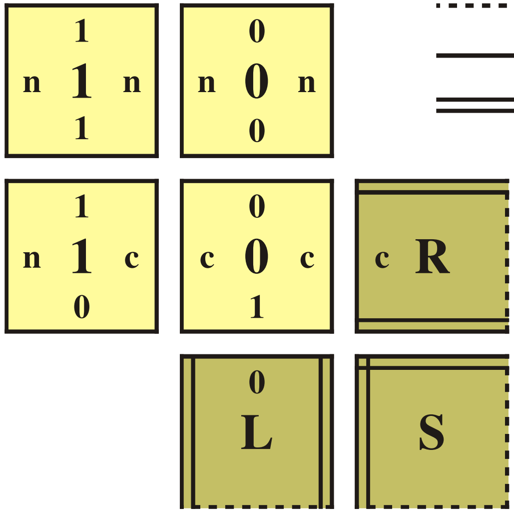

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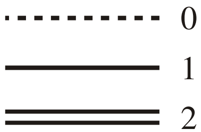

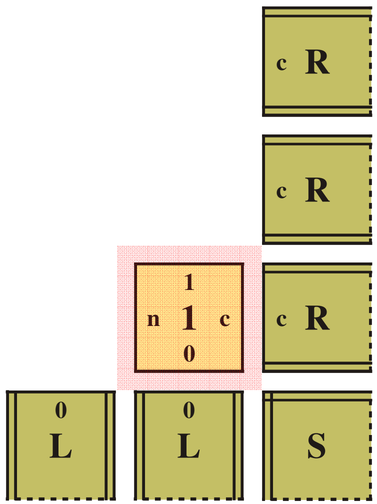

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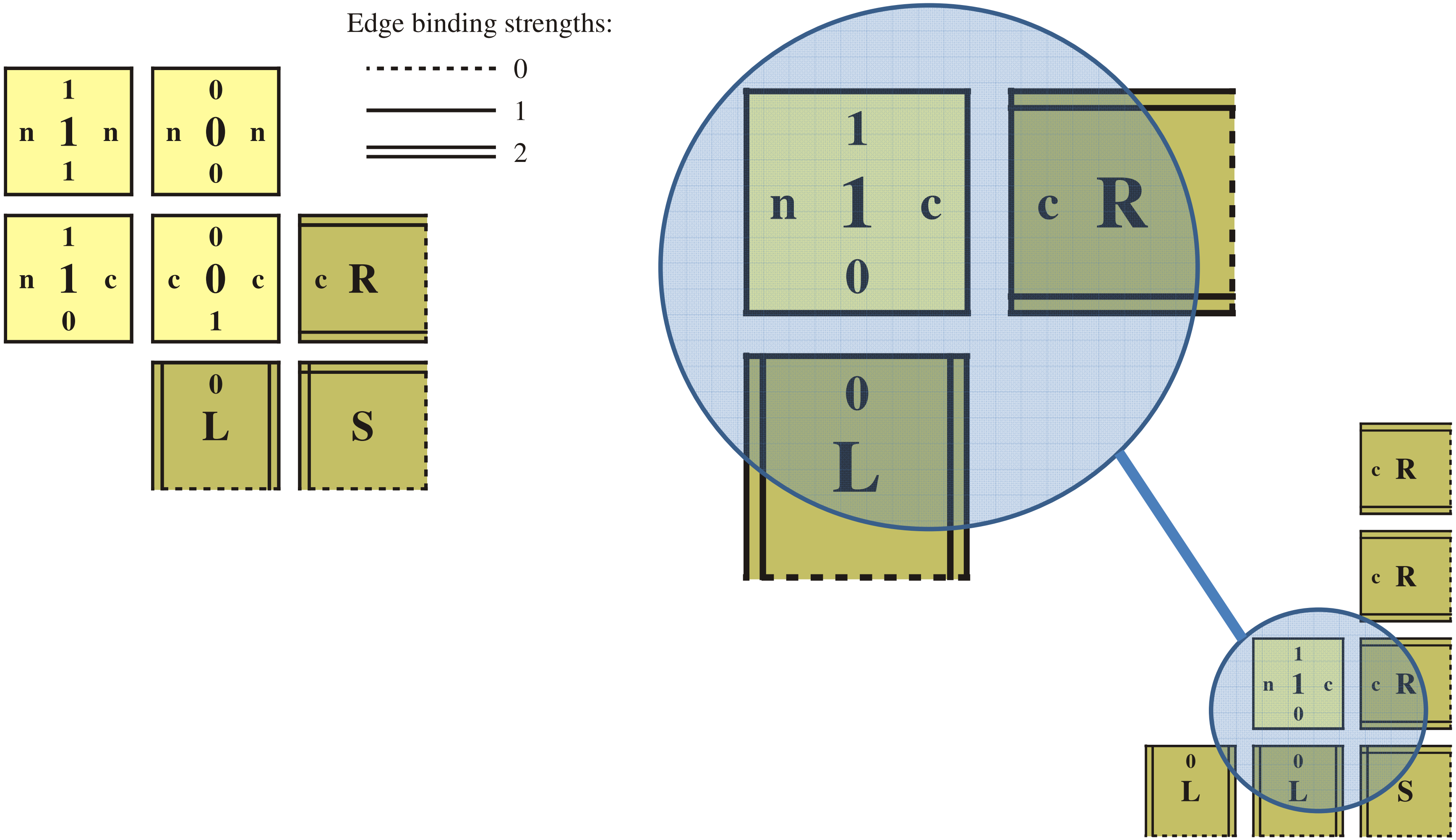

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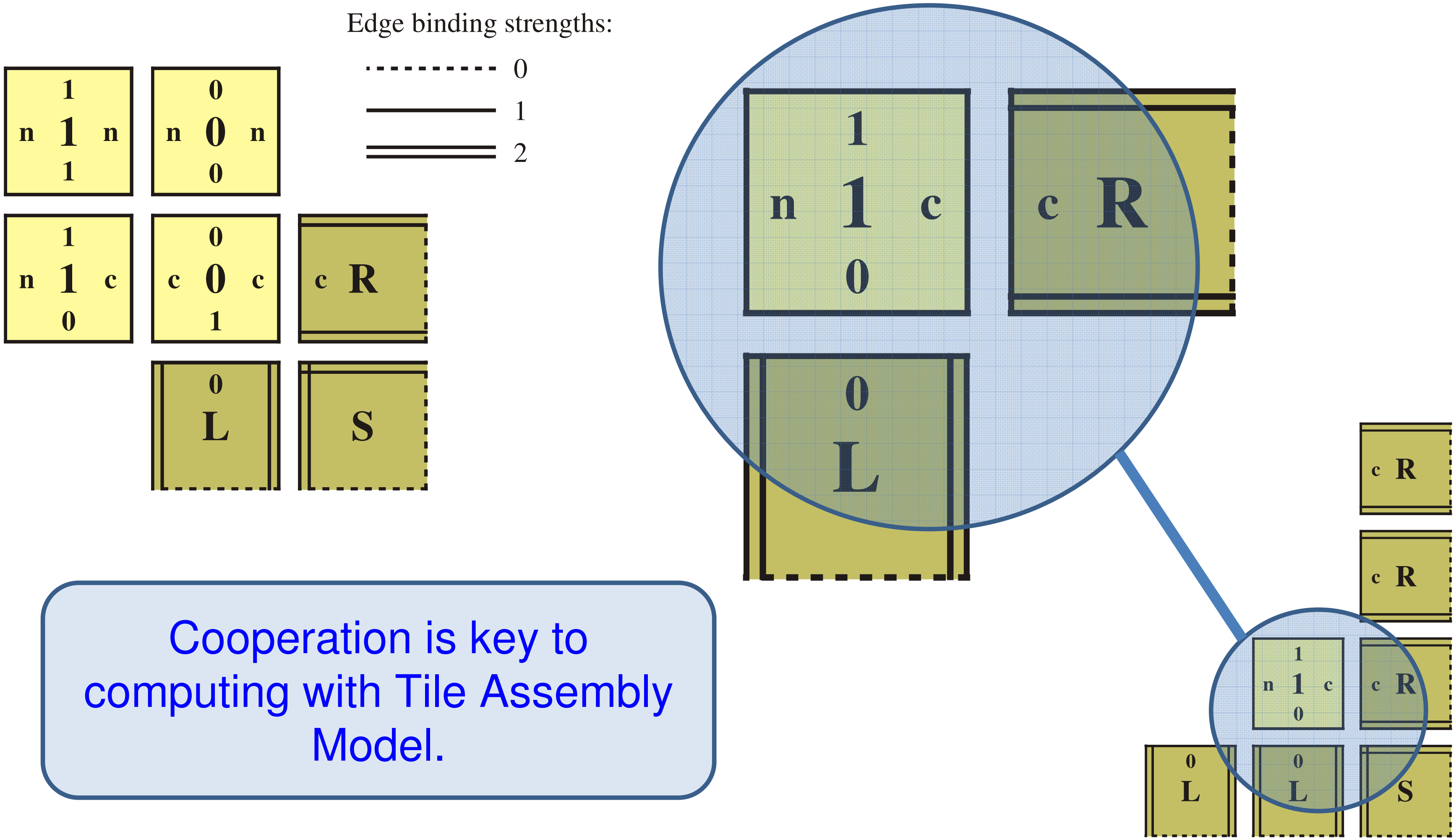
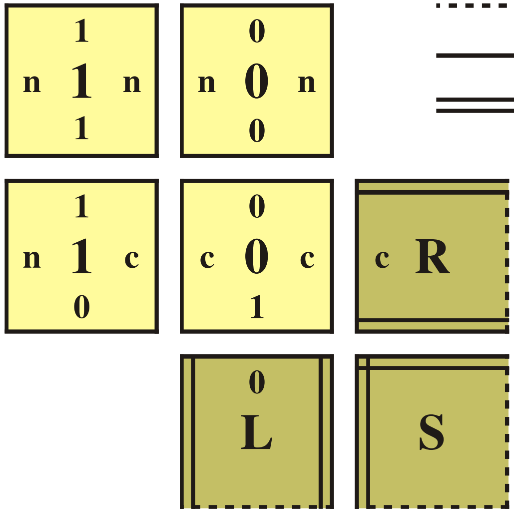
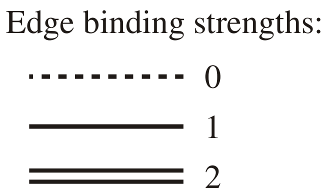
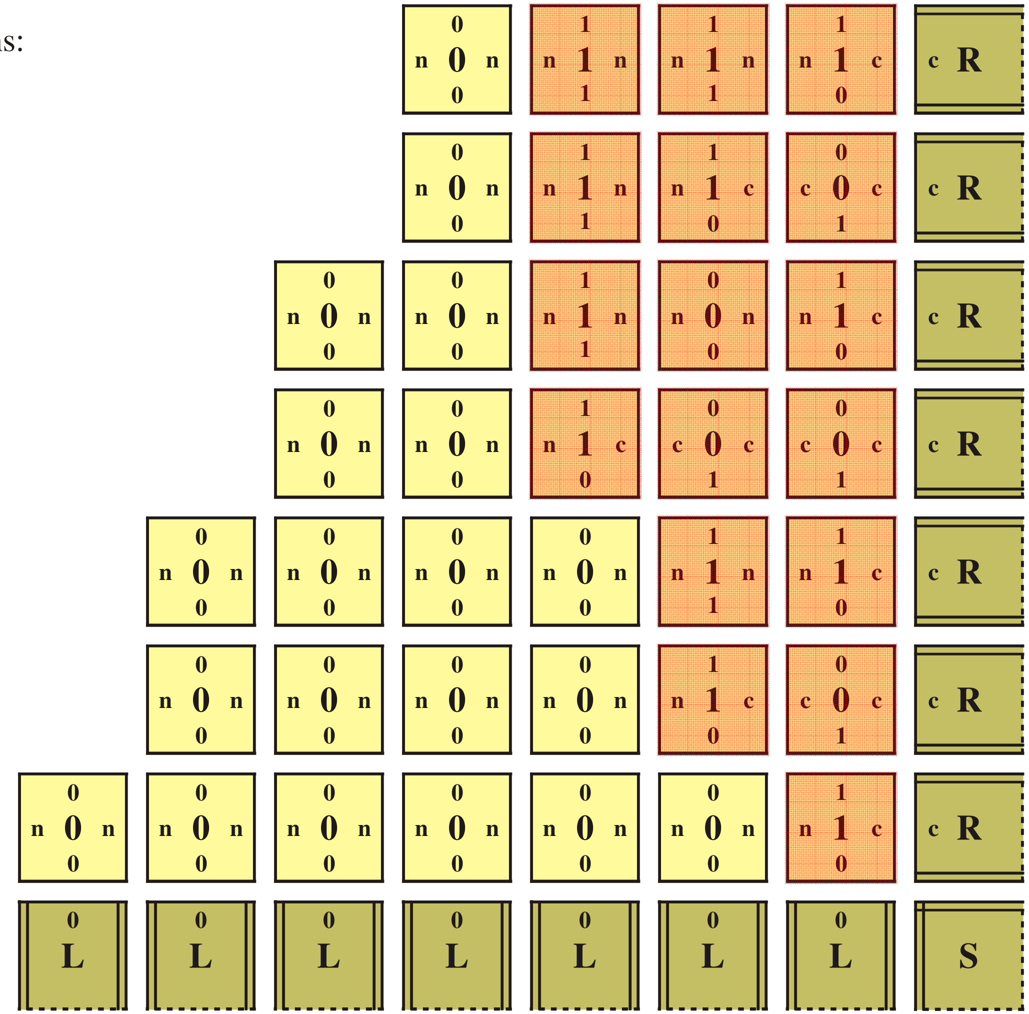


Cooperation is key to computing with Tile Assembly Model.

Tile Assembly Example







Our Results

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First we'll define two supplementary constructions…

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- Utilizes well known technique of simulating a Turing machine with a tile assembly
  - Every row of the assembly encodes the entire configuration of the Turing machine (tape and state) at a particular step in the computation
  - The assembly simulates a ‘one-way infinite to the right’ tape by adding a tile on the right side of the row for each subsequent computation step

Wedge Construction Example

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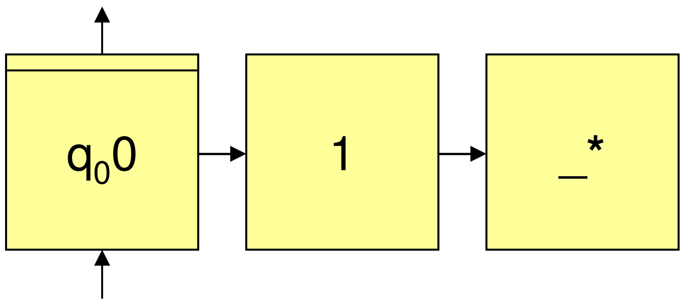
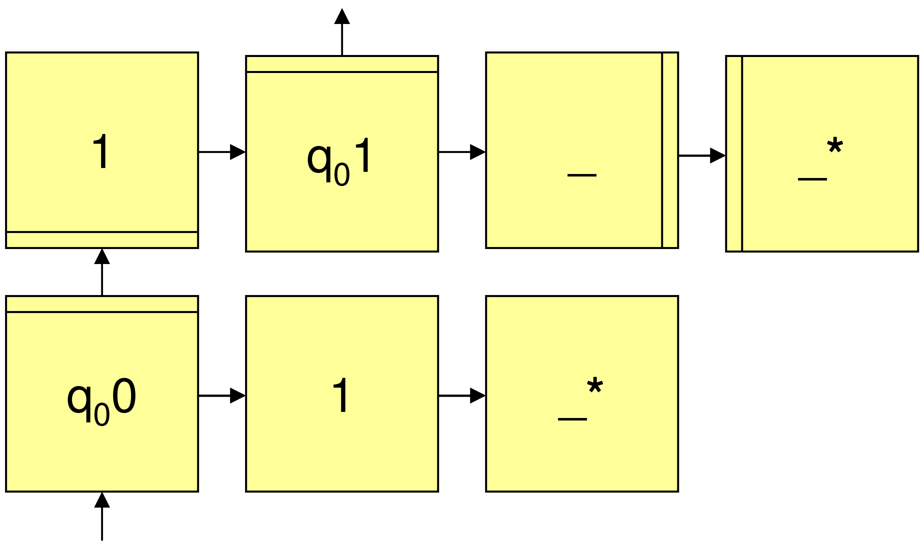


Initial configuration: TM on input ’01’

Wedge Construction Example



After 1 computation step

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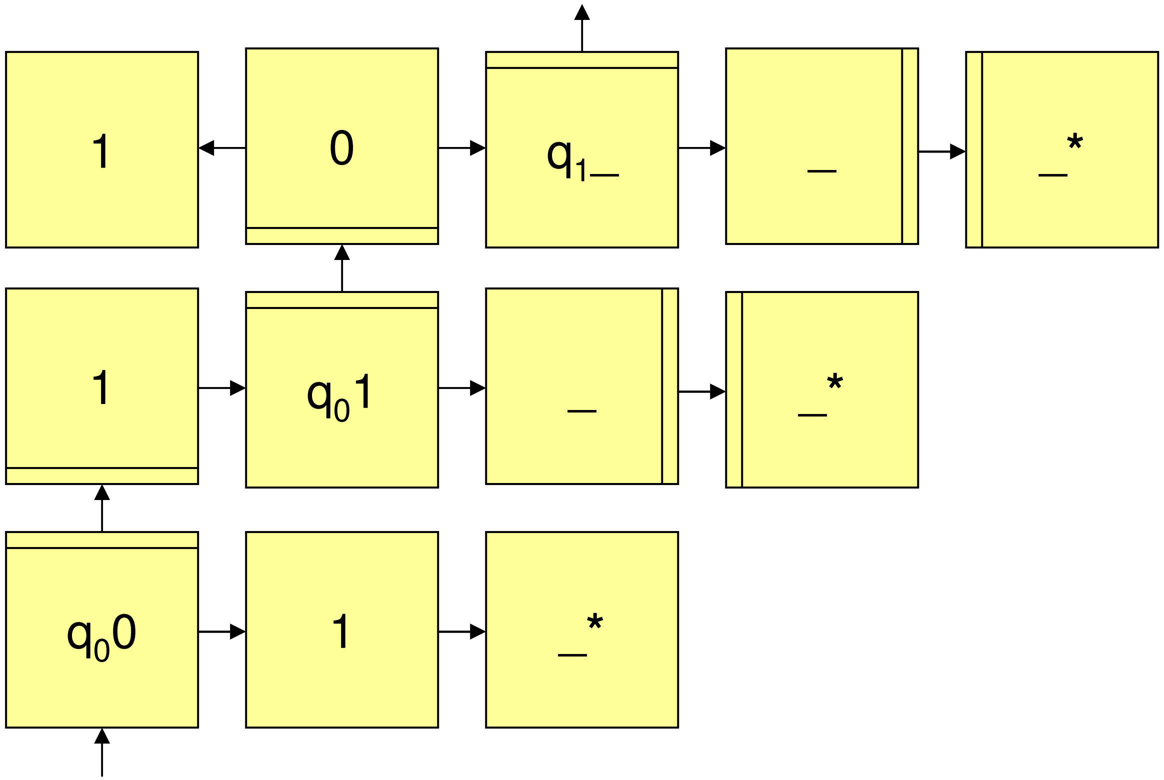

After 2 computation steps

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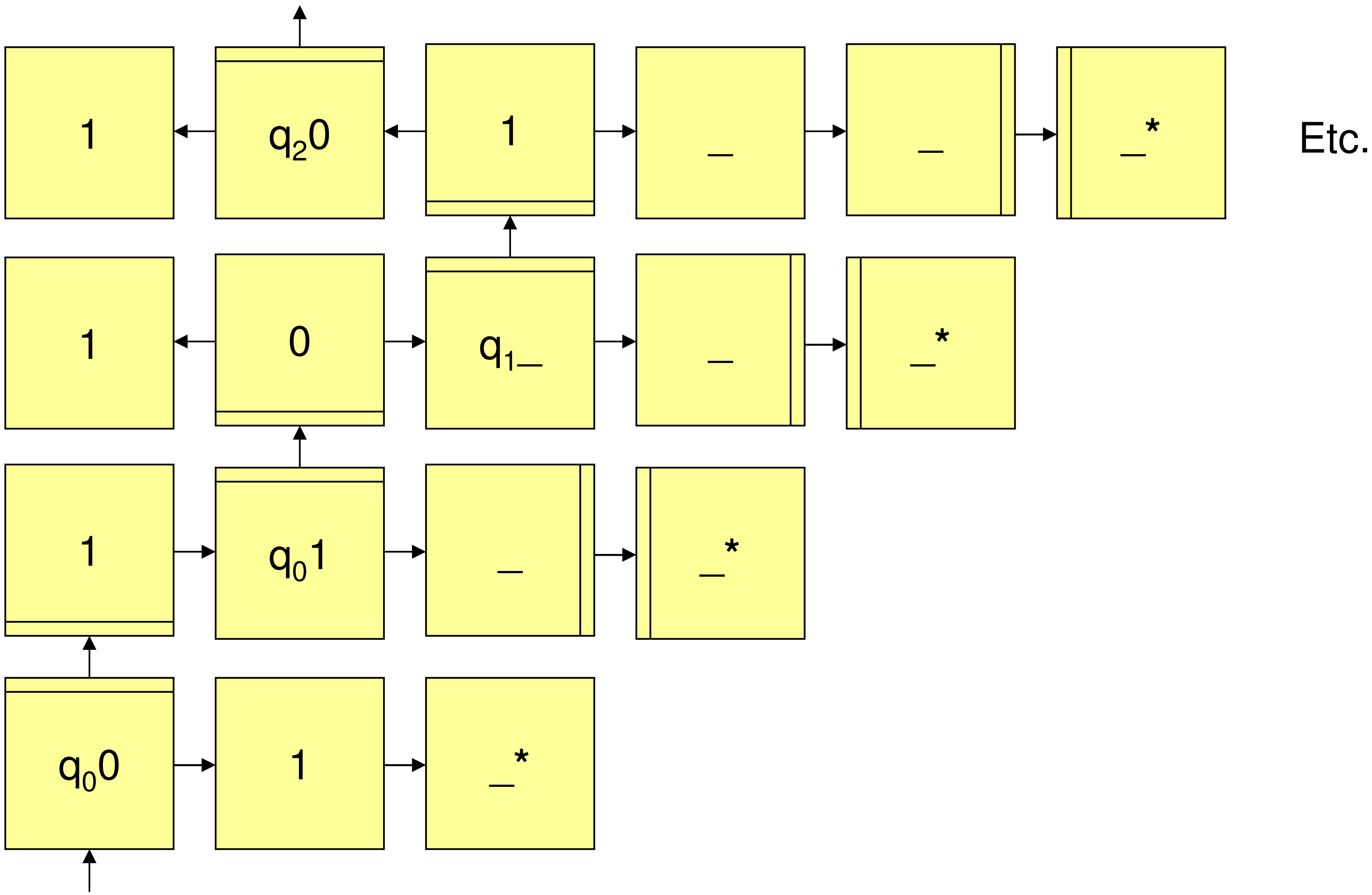


Log-Width Binary Counter

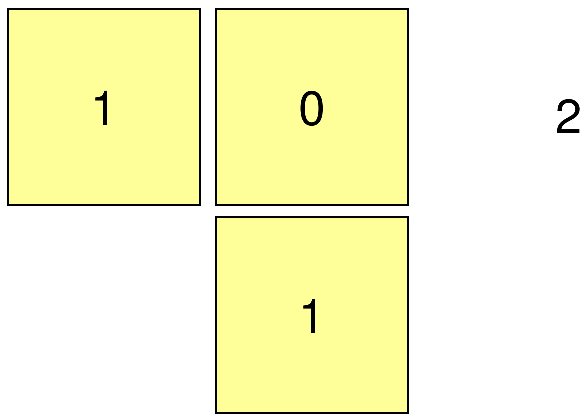
- An assembly which simulates a binary counter from 0 to infinity as it grows upward
- Each row represents a single value, which is one greater than the value of the row beneath it
- The width of each row is equal to the (floor of) the $\log_2$ of the counter's value in that row

Log-Width Binary Counter Example

1

1

Log-Width Binary Counter Example



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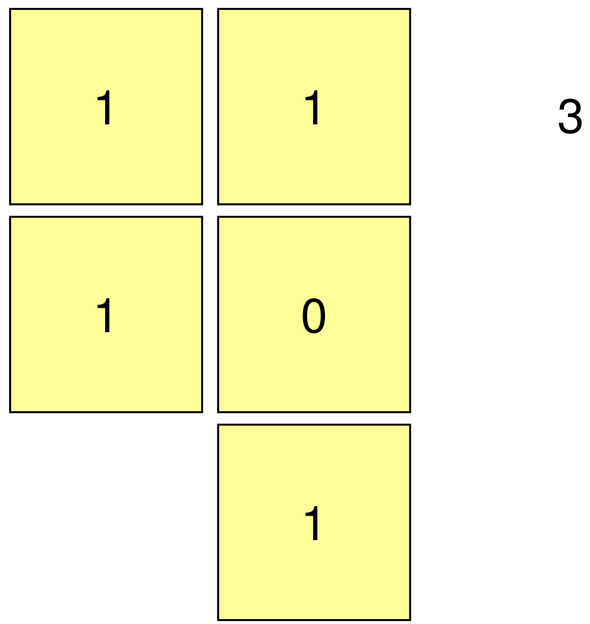

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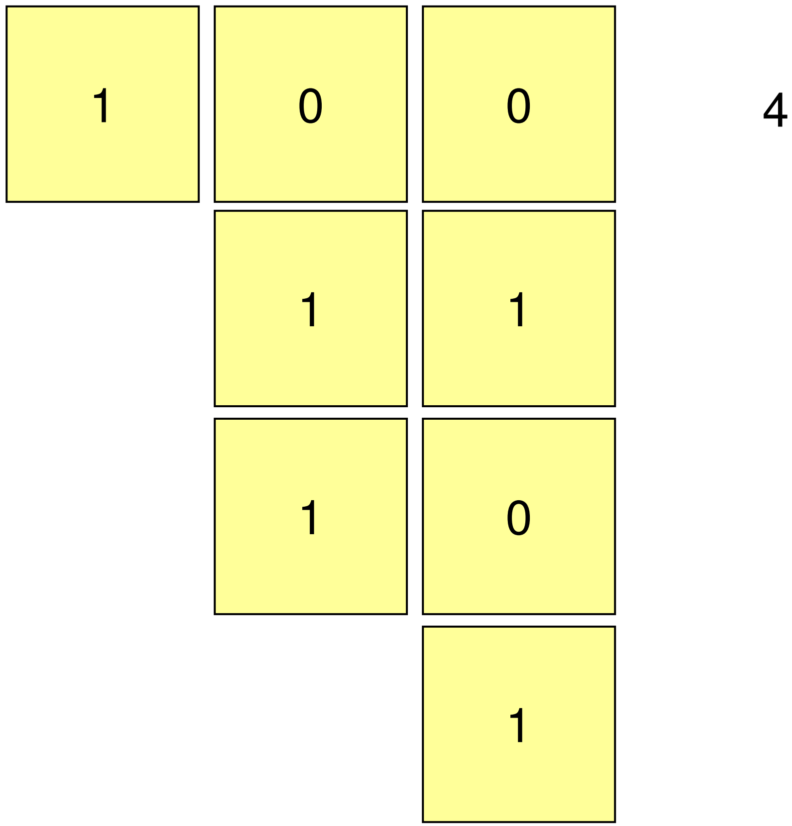

Log-Width Binary Counter Example

1	0	0	0	0
	1	1	1	1
	1	1	1	0
	1	1	0	1
	1	1	0	0
	1	0	1	1
	1	0	1	0
	1	0	0	1
	1	0	0	0
		1	1	1
		1	1	0
		1	0	1
		1	0	0
			1	1
			1	0
				1

A New Characterization of Decidable Languages

Theorem: Let $A \subseteq \mathbf{N}$. The set A is decidable if and only if $A \times \{0\}$ and $A^c \times \{0\}$ weakly self-assemble.

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Proof: ($\rightarrow$) This direction of our proof consists of a construction that demonstrates the claim.

Construction Overview

Fact: If *A* is decidable, then there exists a TM *M* which halts on every input and accepts exactly those that are in *A*

Construction Overview

while $0 \le n < \infty$ do

  simulate M on the binary representation of n

  if M accepts, then

    output 1

  else

    output 0

  end if

  $n := n + 1$

end while

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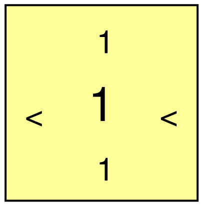
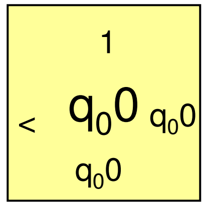
Incrementing the inputs is done by embedding a log-width binary counter within the wedge constructions

Construction Technique: Embedding Functionality

Log-width binary counter tile

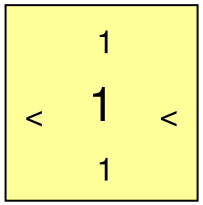


Turing machine tile



Construction Technique: Embedding Functionality

Log-width binary counter tile



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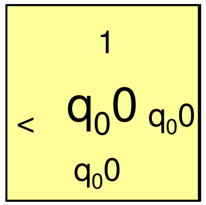

Turing machine tile passing binary counter value upward

Construction Technique: Embedding Functionality

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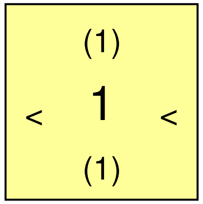
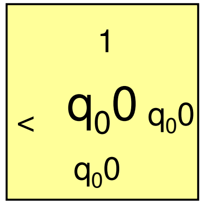
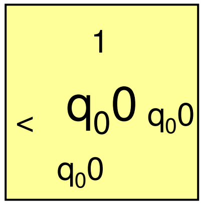


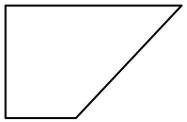
Turing machine tile



Turing machine tile passing binary counter value upward

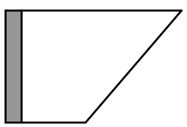


Construction Diagram



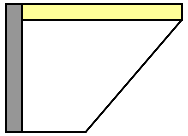
Start with a wedge construction
which simulates TM *M* on input 0

Construction Diagram



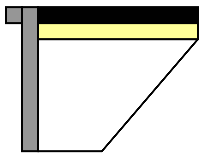
Embed a log-width binary
counter along the left side

Construction Diagram



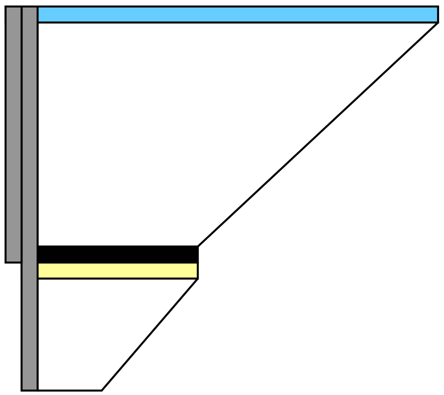
Once $M(0)$ halts and accepts or rejects,
make a row specifying the result

Construction Diagram



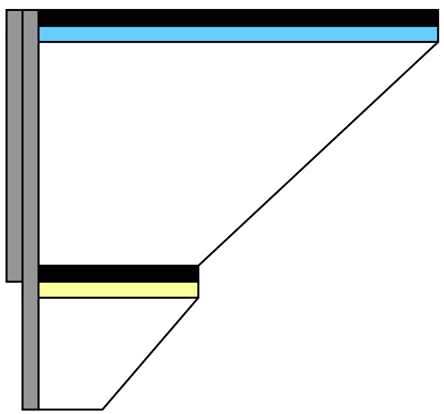
Next, add a row which increments the counter

Construction Diagram



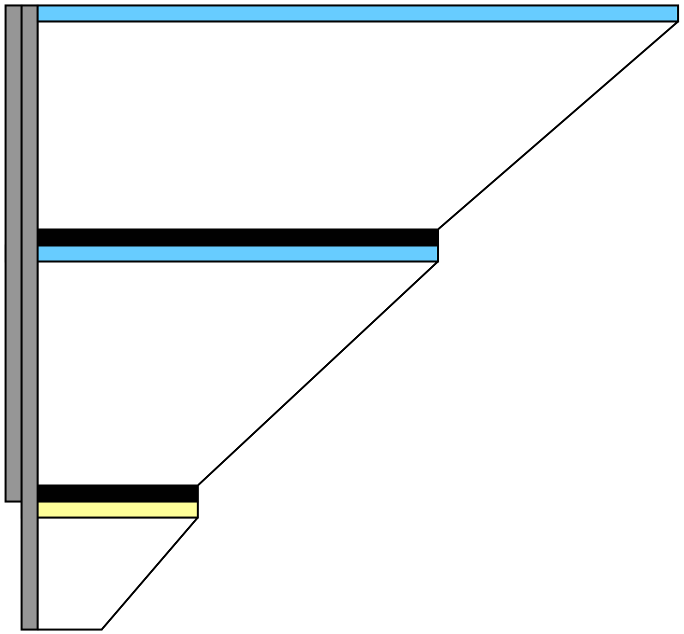
Use the new counter value
to begin the simulation
of *M*(1)

Construction Diagram



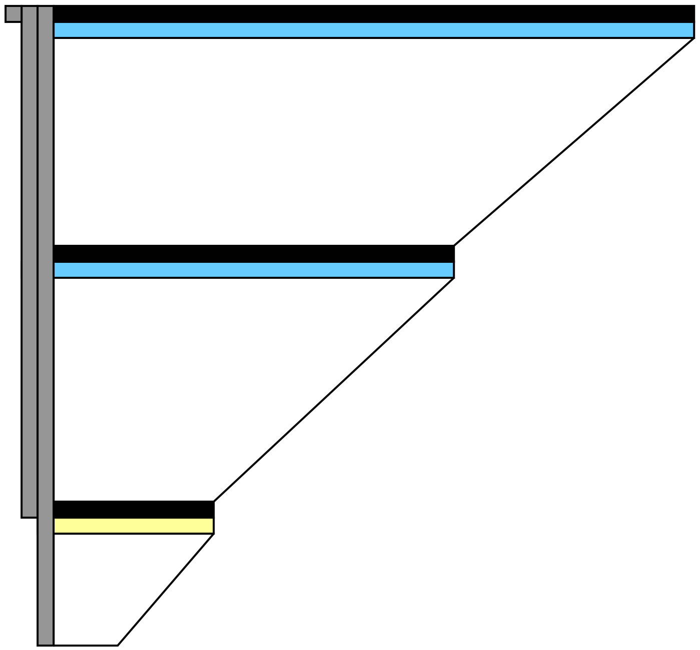
Increment the counter

Construction Diagram



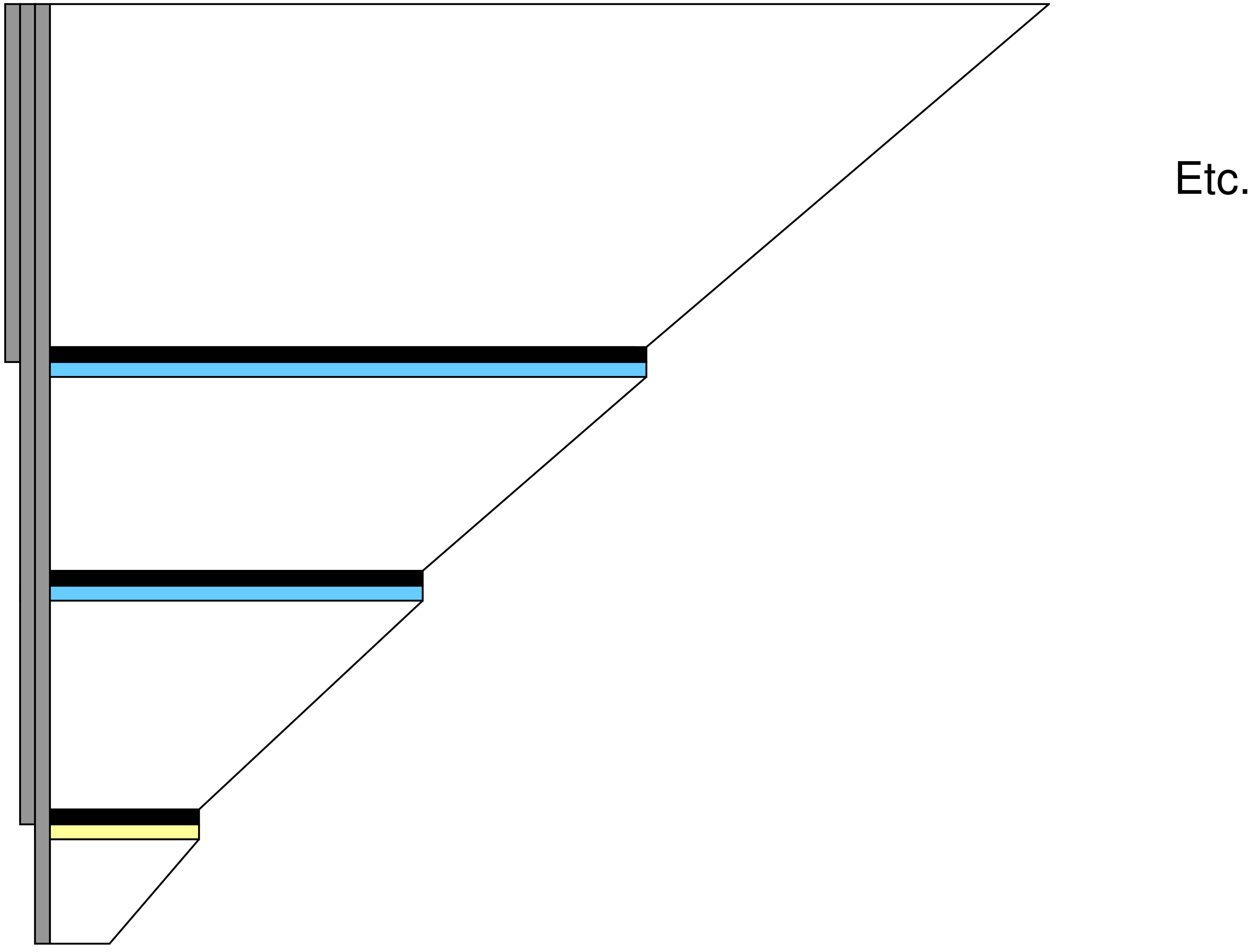
Simulate $M(2)$

Construction Diagram

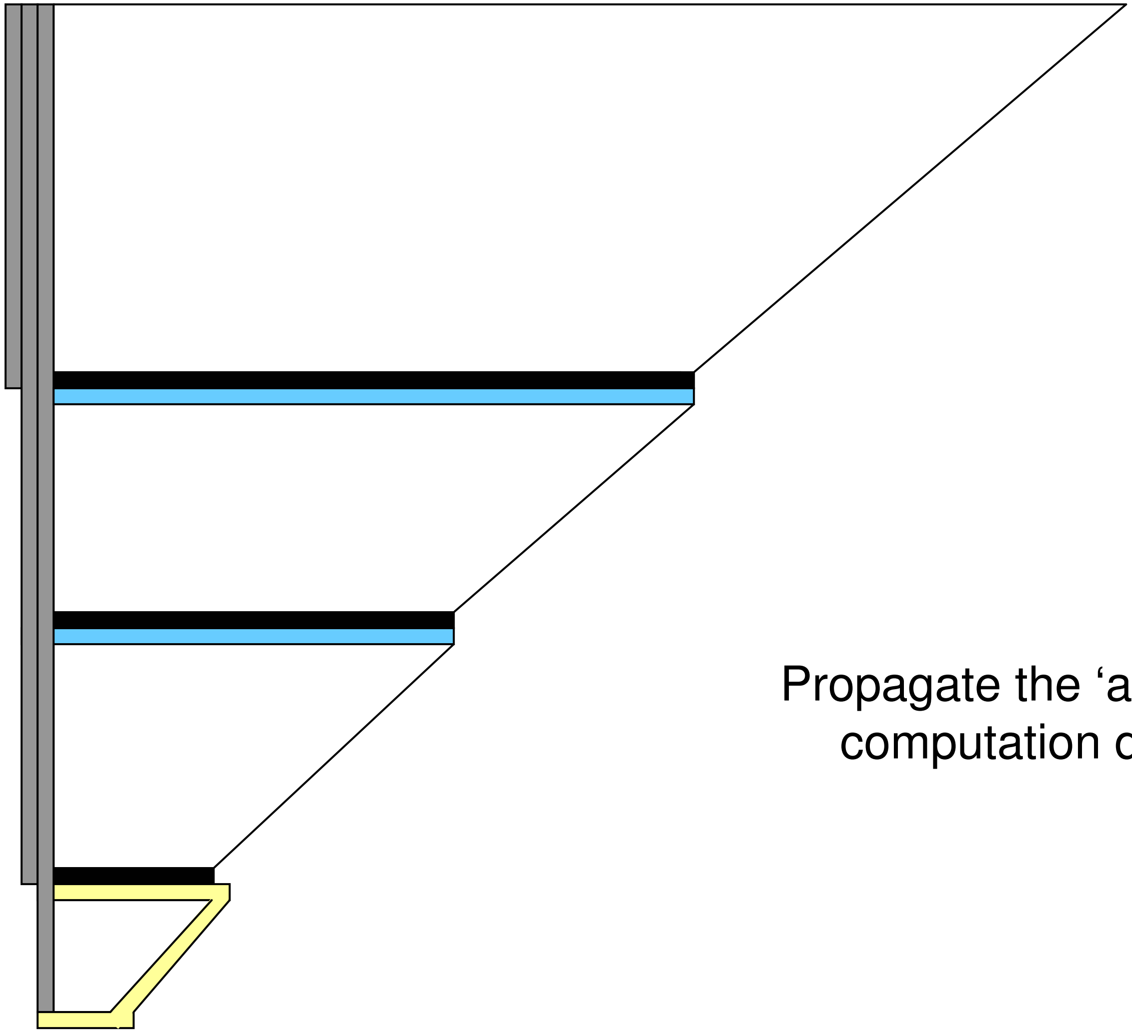


Increment the counter

Construction Diagram

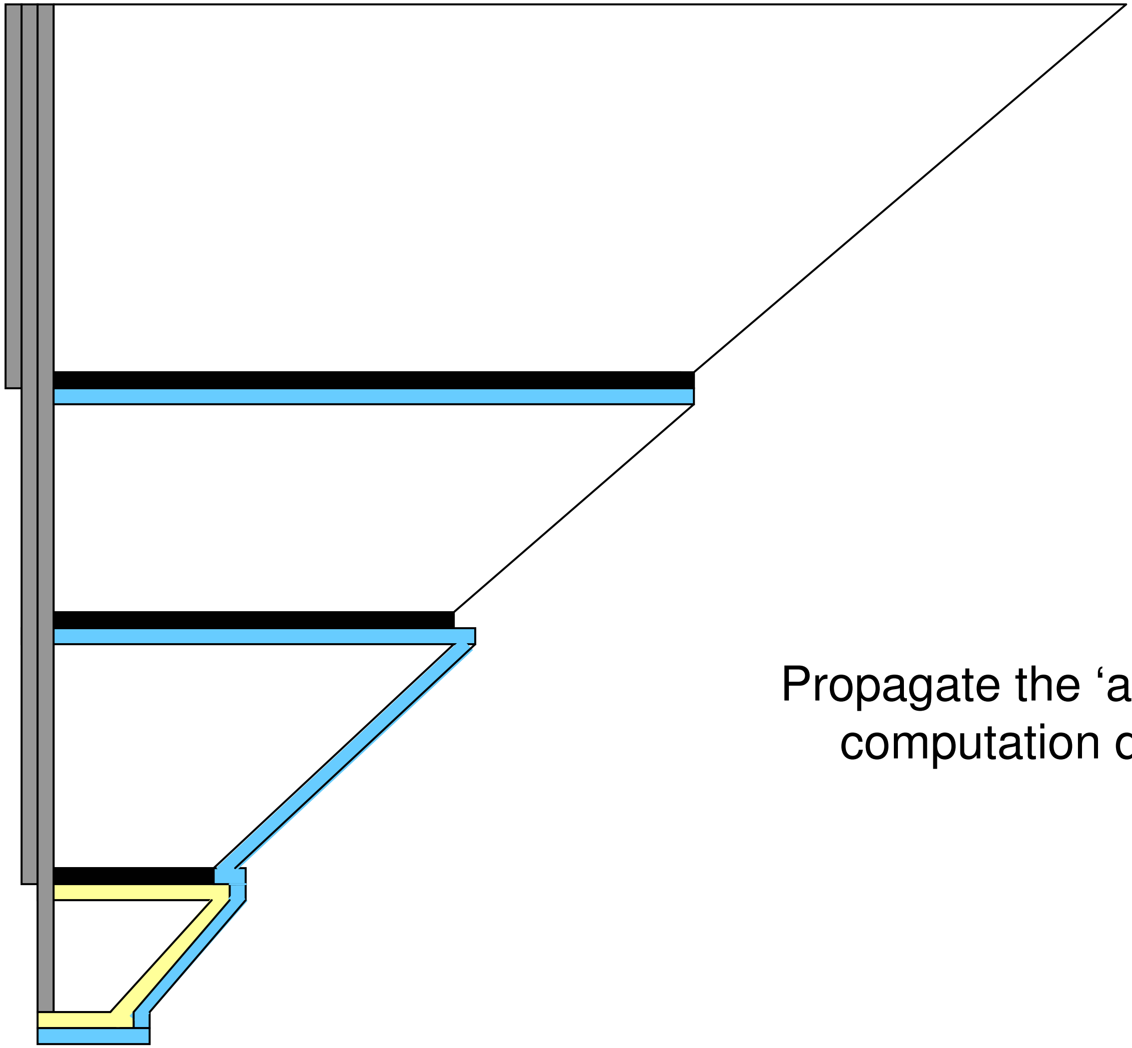


Construction Diagram



Propagate the 'answers' from each computation down to the axis

Construction Diagram



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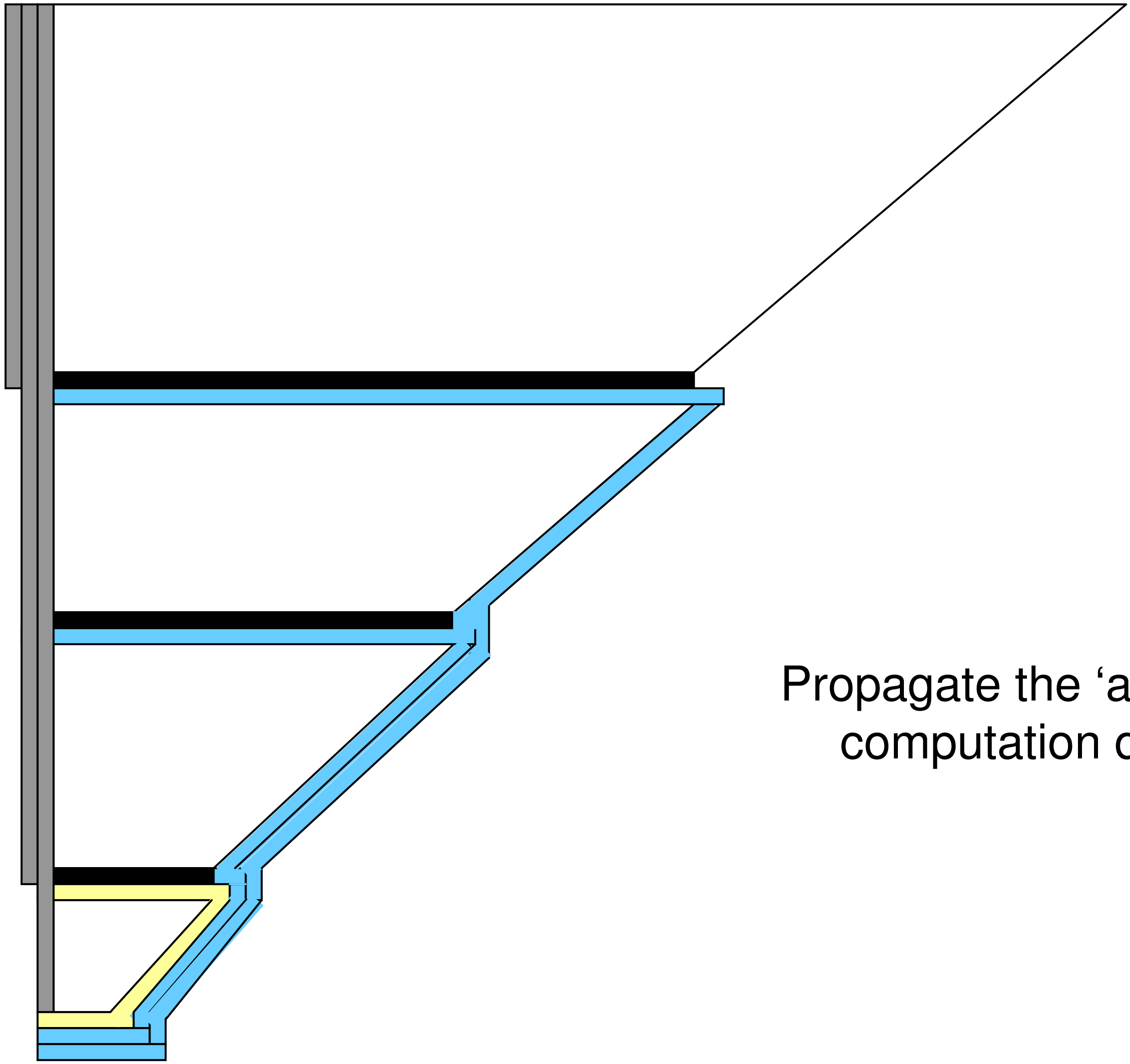
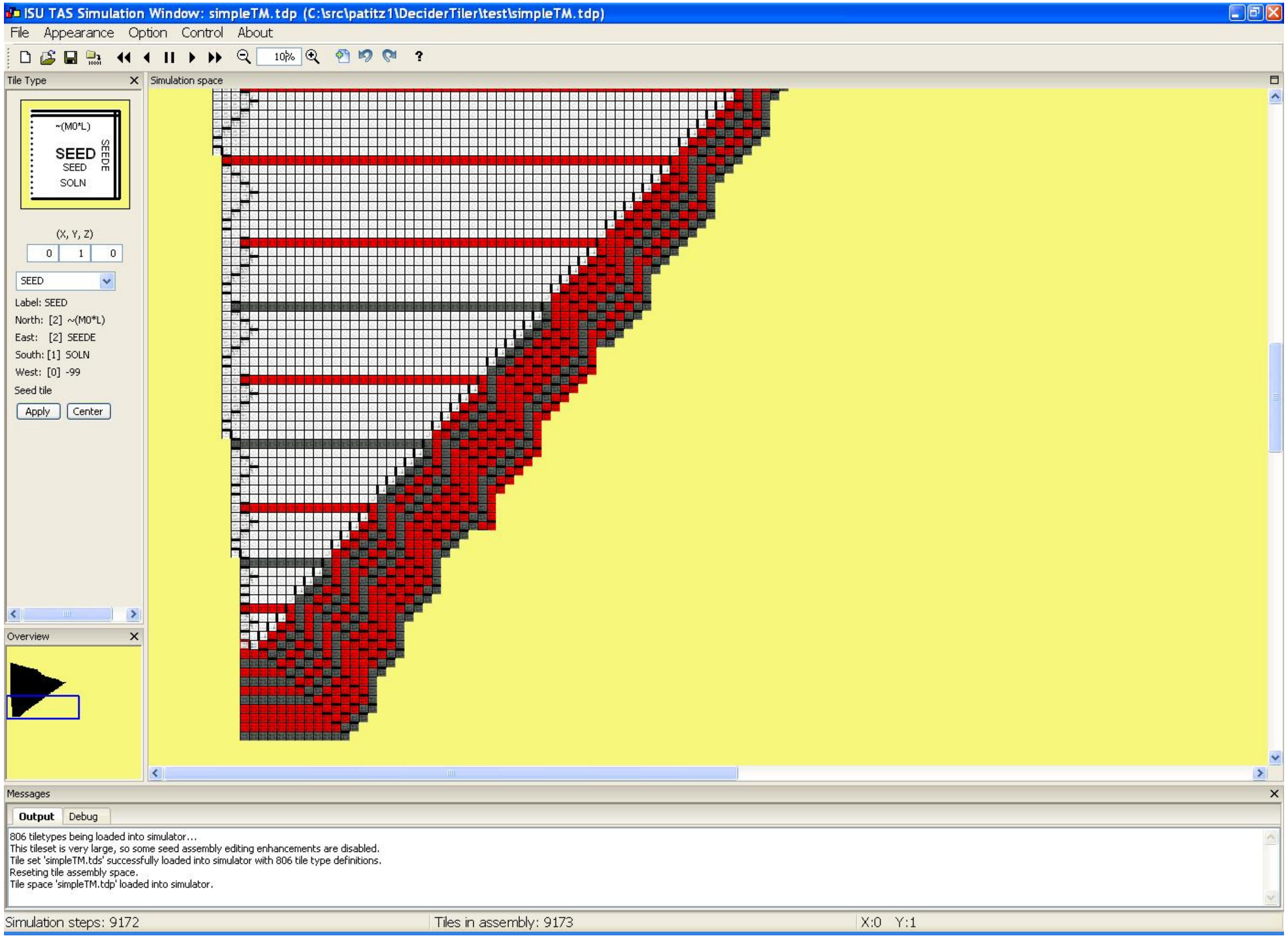


Propagate the ‘answers’ from each computation down to the axis

Construction Example



A New Characterization of Decidable Languages

This construction proves that if a set is decidable, then $A \times \{0\}$ and $A^c \times \{0\}$ weakly self-assemble.

A New Characterization of Decidable Languages

Proof: (⟵) This direction of the proof uses the existence of self-assembly simulators to prove that if $A \times \{0\}$ and $A^c \times \{0\}$ weakly self-assemble, then the set A is decidable.

Proof Sketch (←)

- Assume A and A^c both weakly self-assemble

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- Then there exist tile assembly systems $\mathcal{T}_{A\times\{0\}}$ and $\mathcal{T}_{A^c\times\{0\}}$ in which they weakly self-assemble

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- For an input n, simulate both tile assembly systems in parallel
  - Accept if $\mathcal{T}_{A\times\{0\}}$ puts a black tile at $(n,0)$
  - Reject if $\mathcal{T}_{A^c\times\{0\}}$ puts a black tile at $(n,0)$

A New Characterization of Decidable Languages

This completes the proof that a set A is decidable if and only if $A \times \{0\}$ and $A^c \times \{0\}$ weakly self-assemble.

Second Main Result

To prove our first main result, we constructed a tile assembly system that placed at least one tile in three different quadrants.

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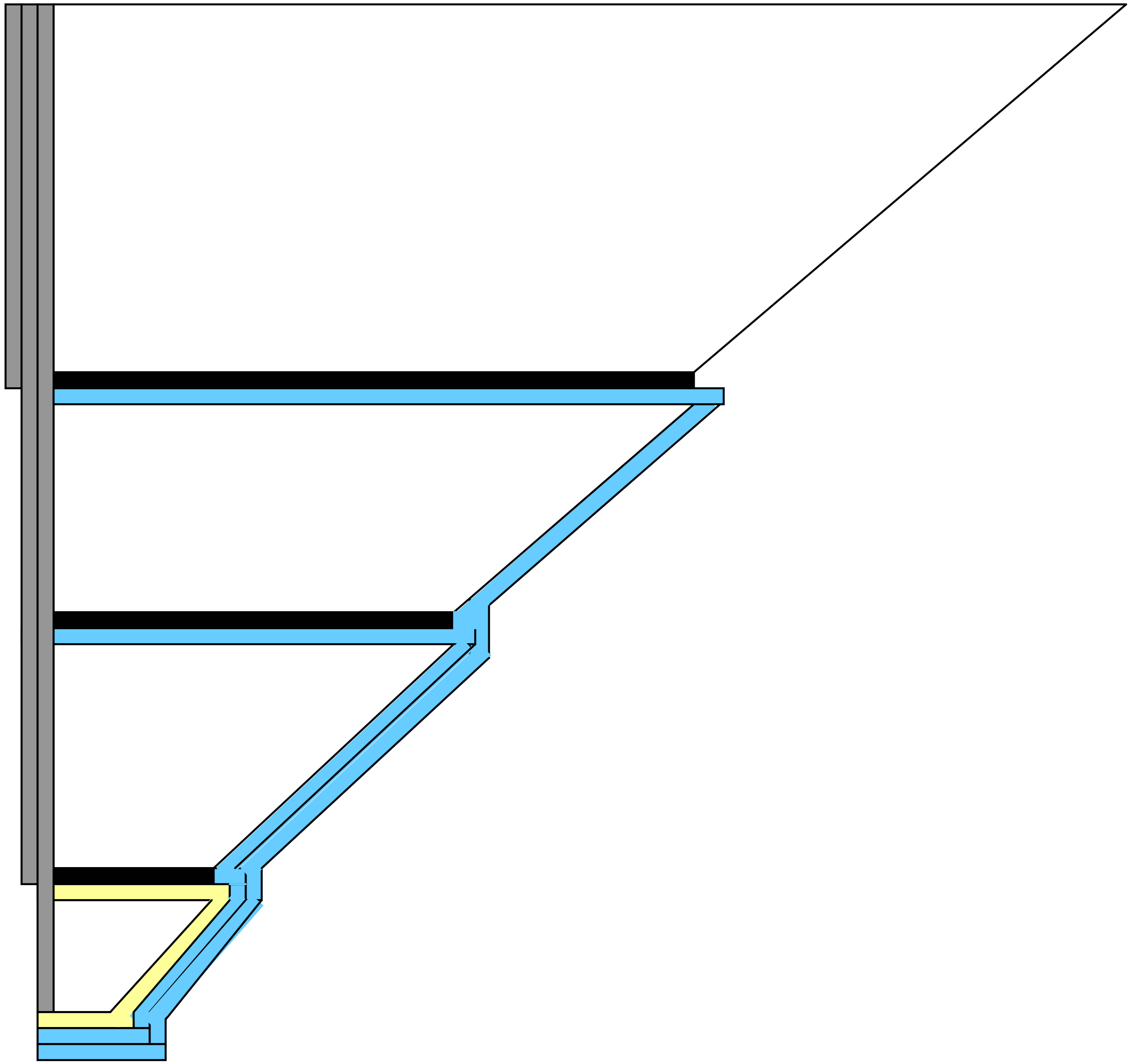
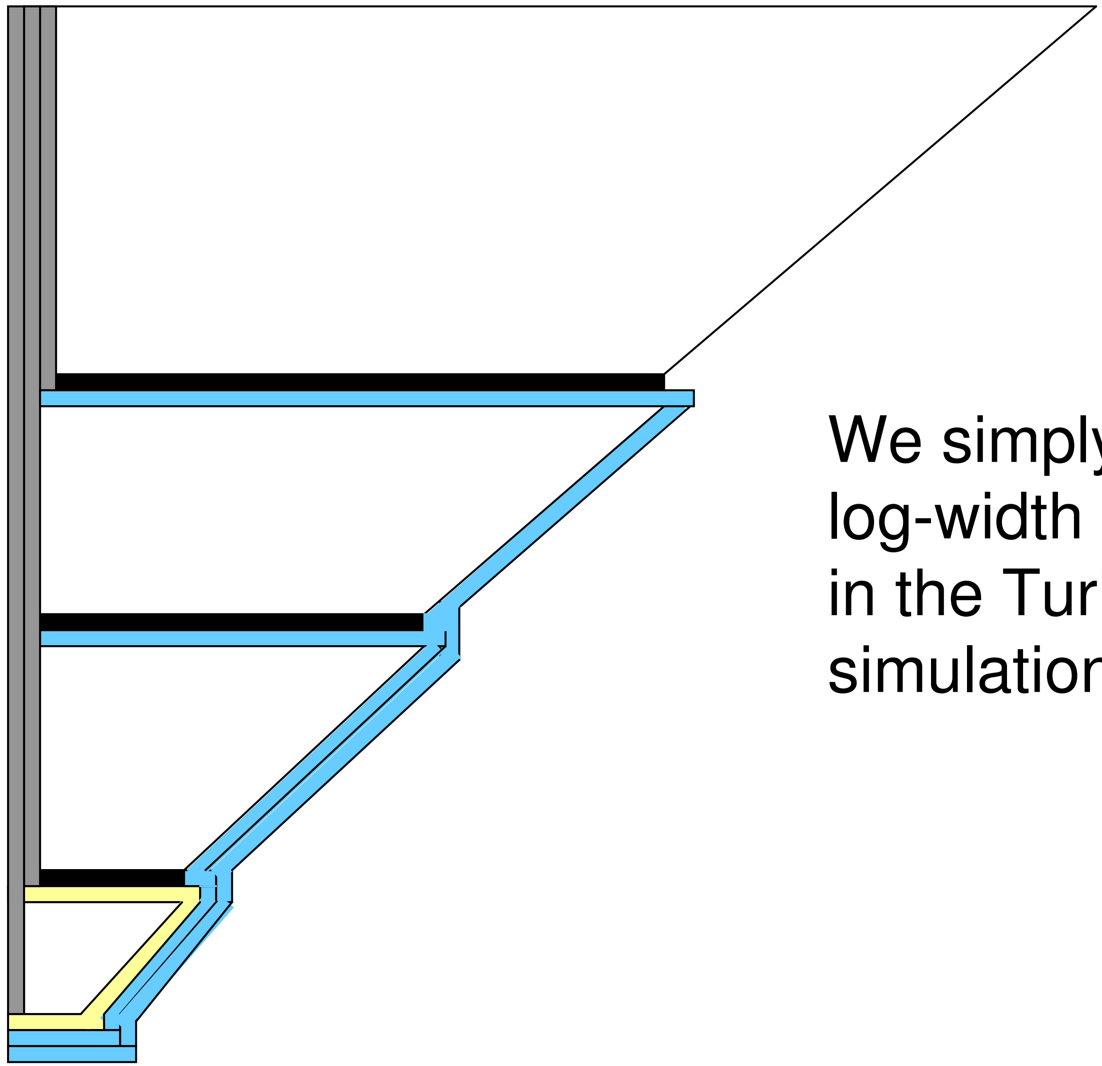
Note that it *is* possible to prove our first main result while placing tiles in only **two quadrants**.

Two quadrants



Two quadrants



We simply embed the log-width binary counter in the Turing machine simulation.

Second Main Result

However, if the language *A* has sufficient space complexity, **AND** your tile assembly system *resembles* our construction in the sense that the TM is simulated “row by row,” then two quadrants of space are **necessary**.

Second Main Result (proof idea)

Assumption #1: connected to each point along the *x*-axis is a *unique* longest path originating from some unique point in the first quadrant that carries the answer to the question:

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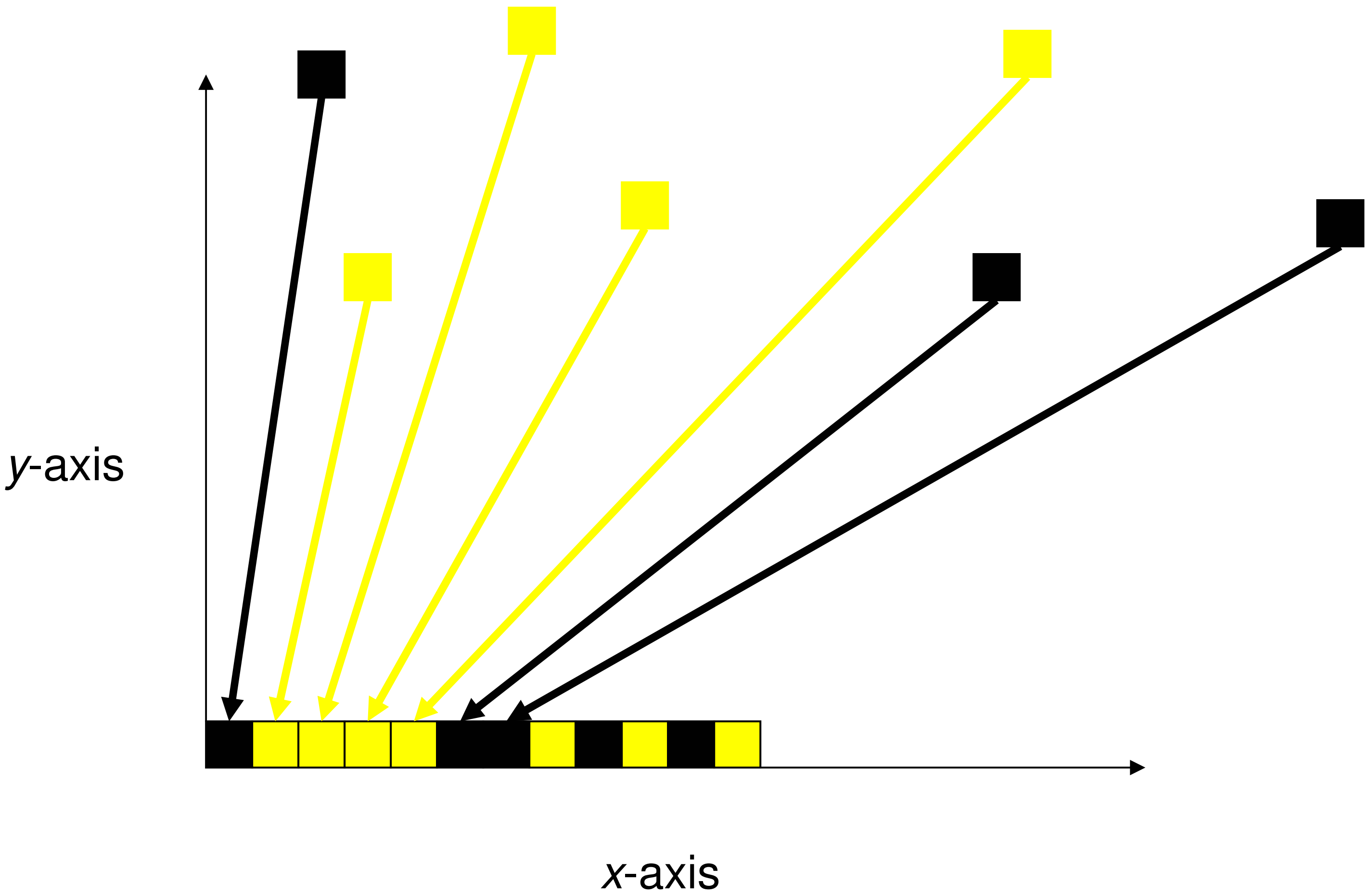
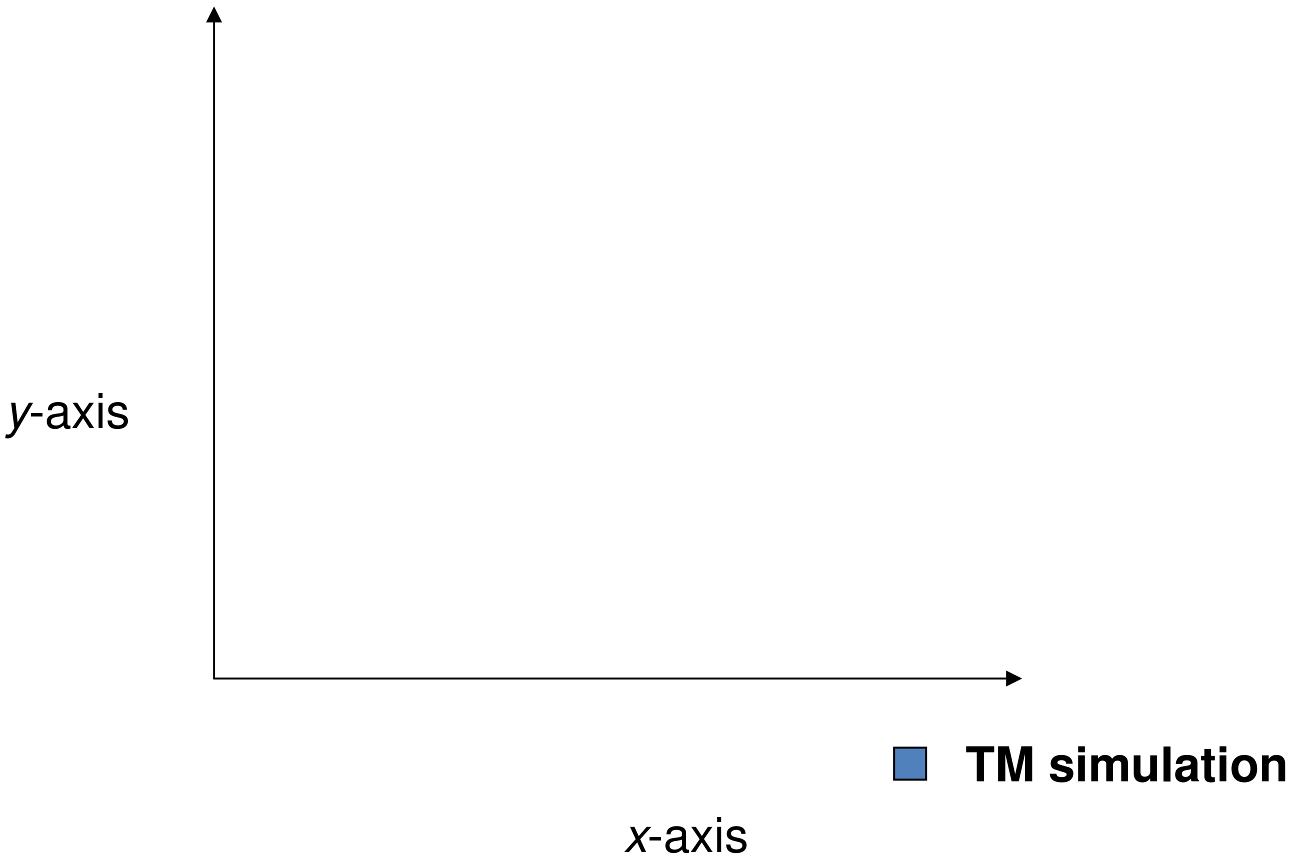
Second Main Result (proof idea)



Second Main Result (proof idea)

Assumption #2: aside from all of the paths mentioned on the previous slide, the rest of the assembly can be self-assembled (entirely) one row at a time

Second Main Result (proof idea)



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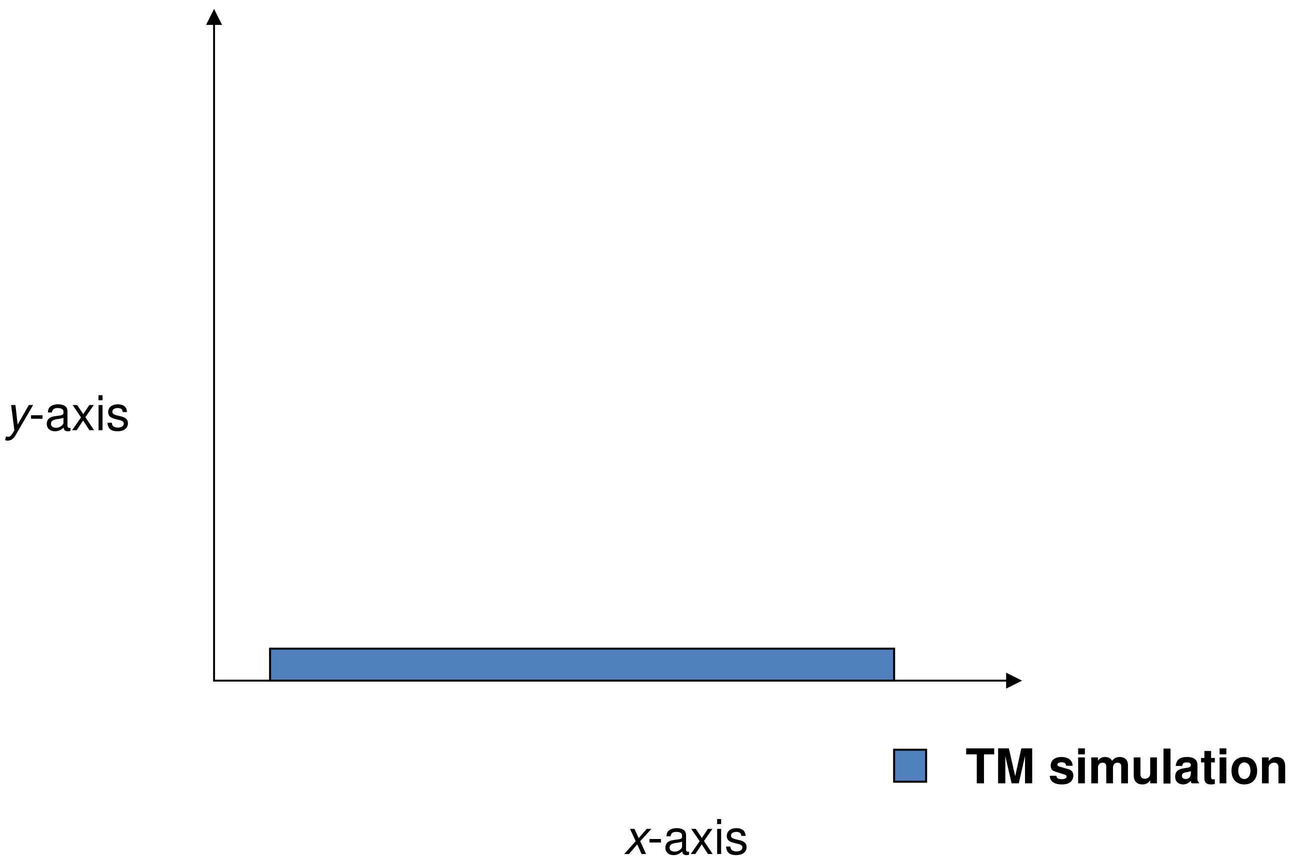

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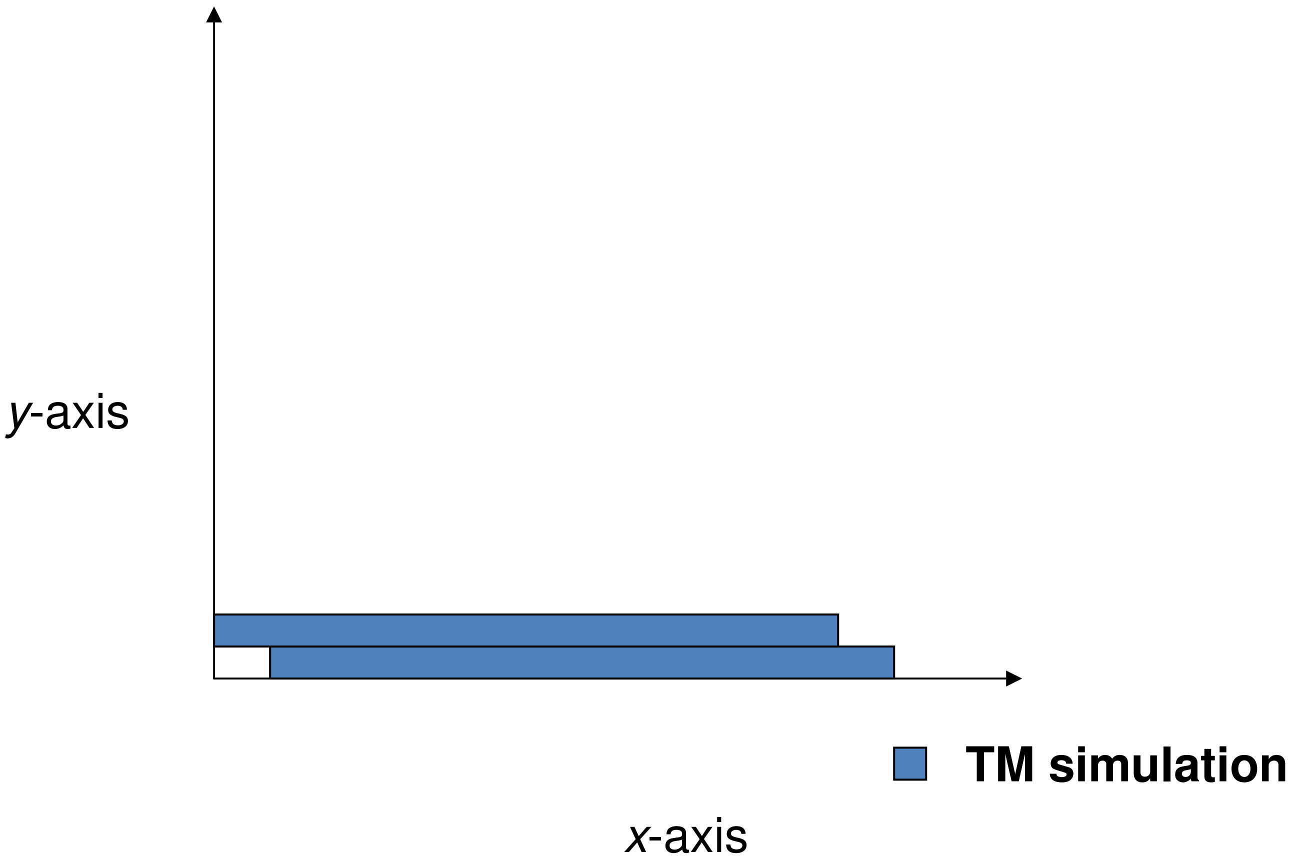

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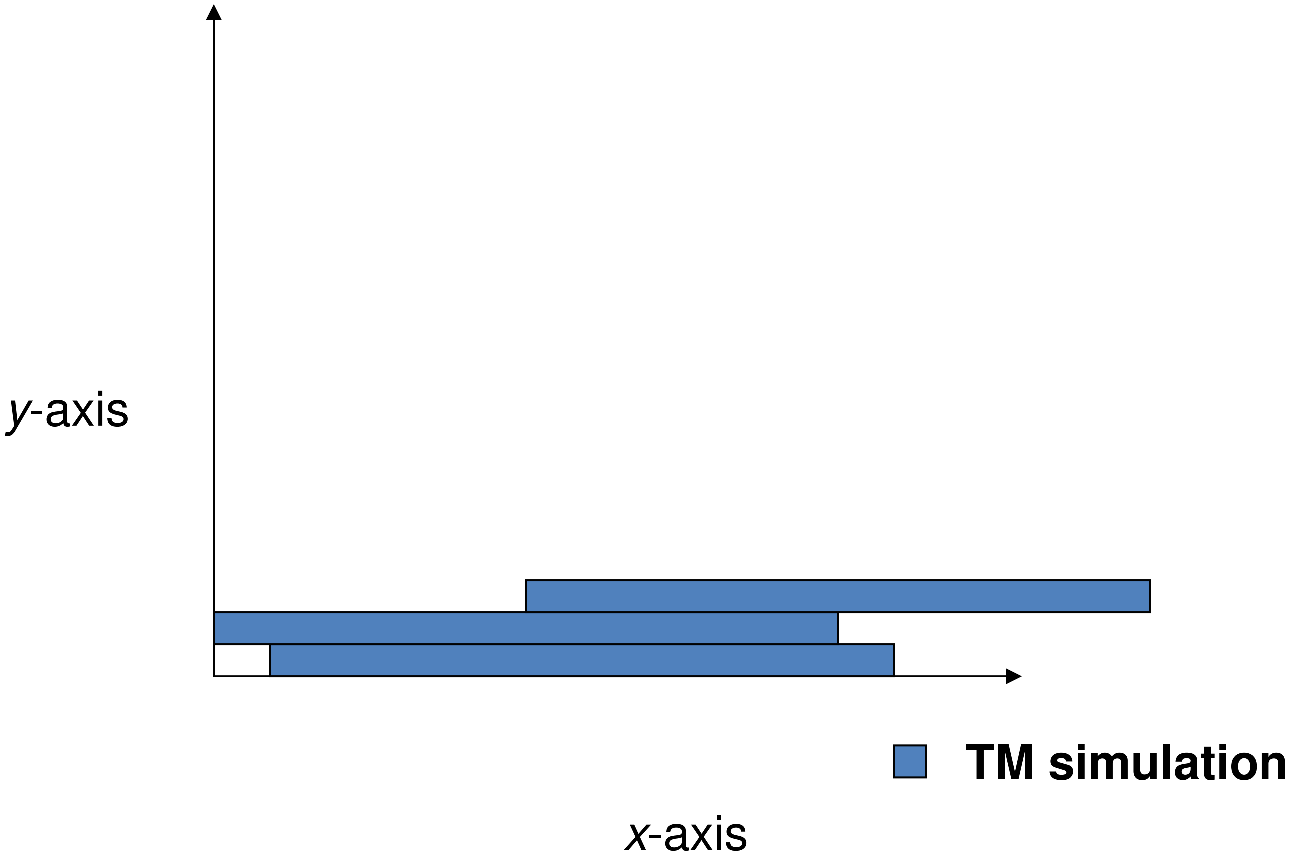

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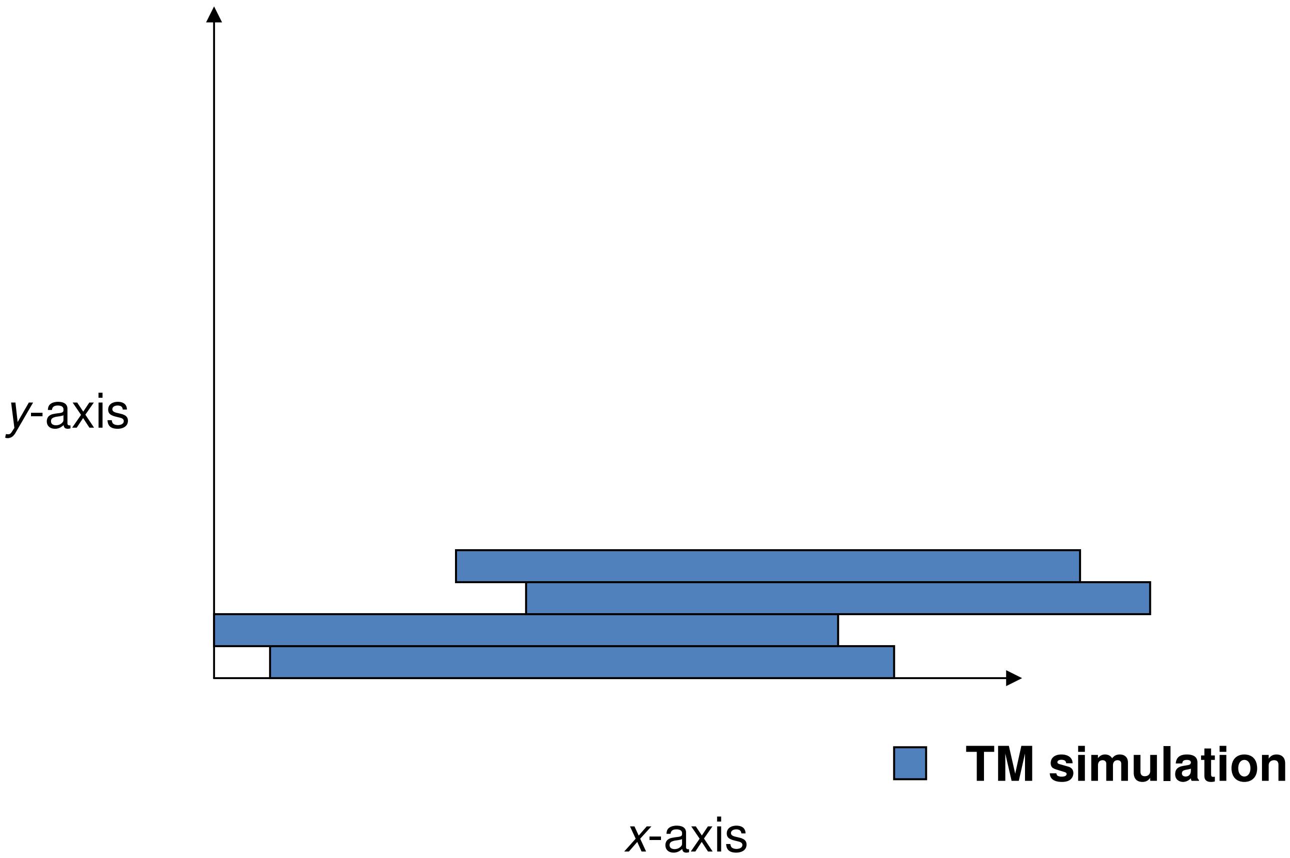

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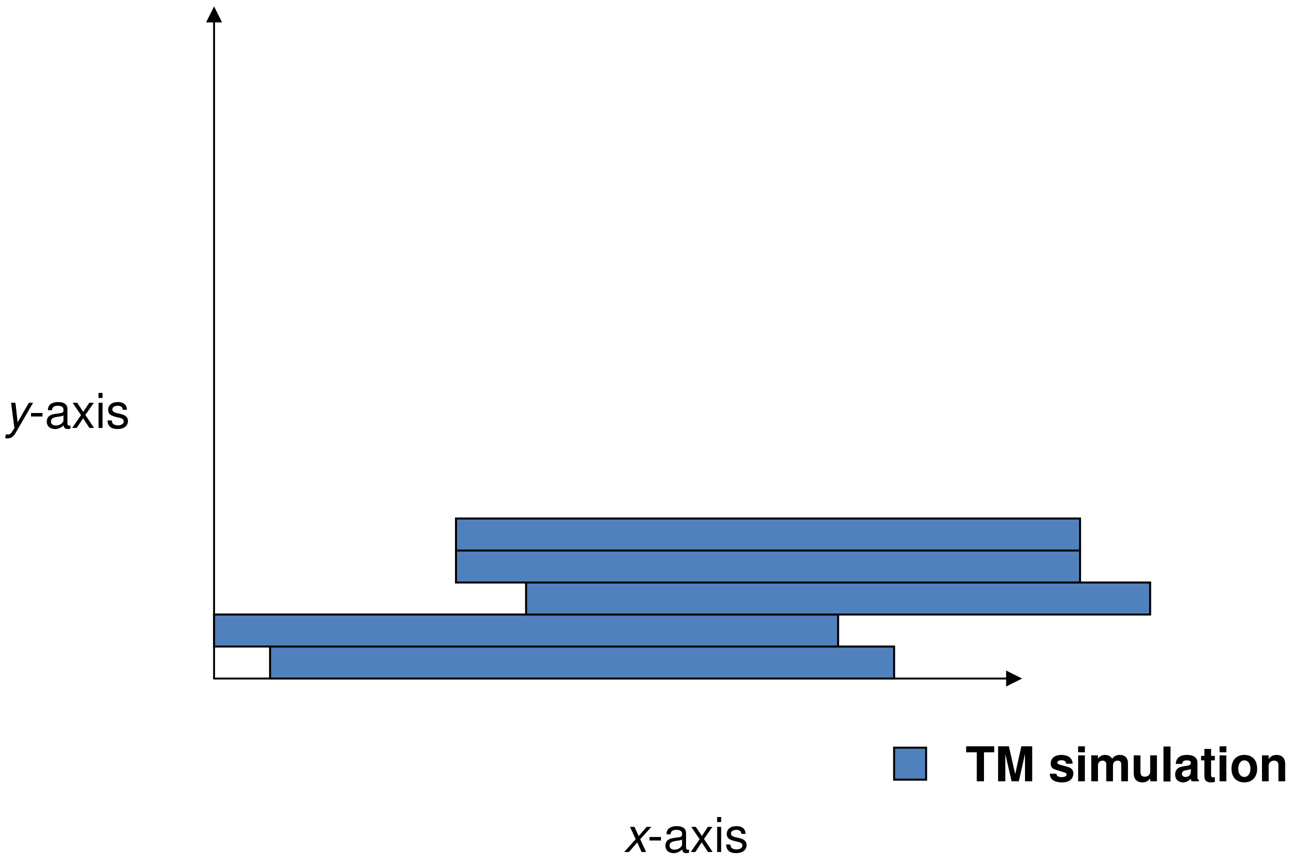

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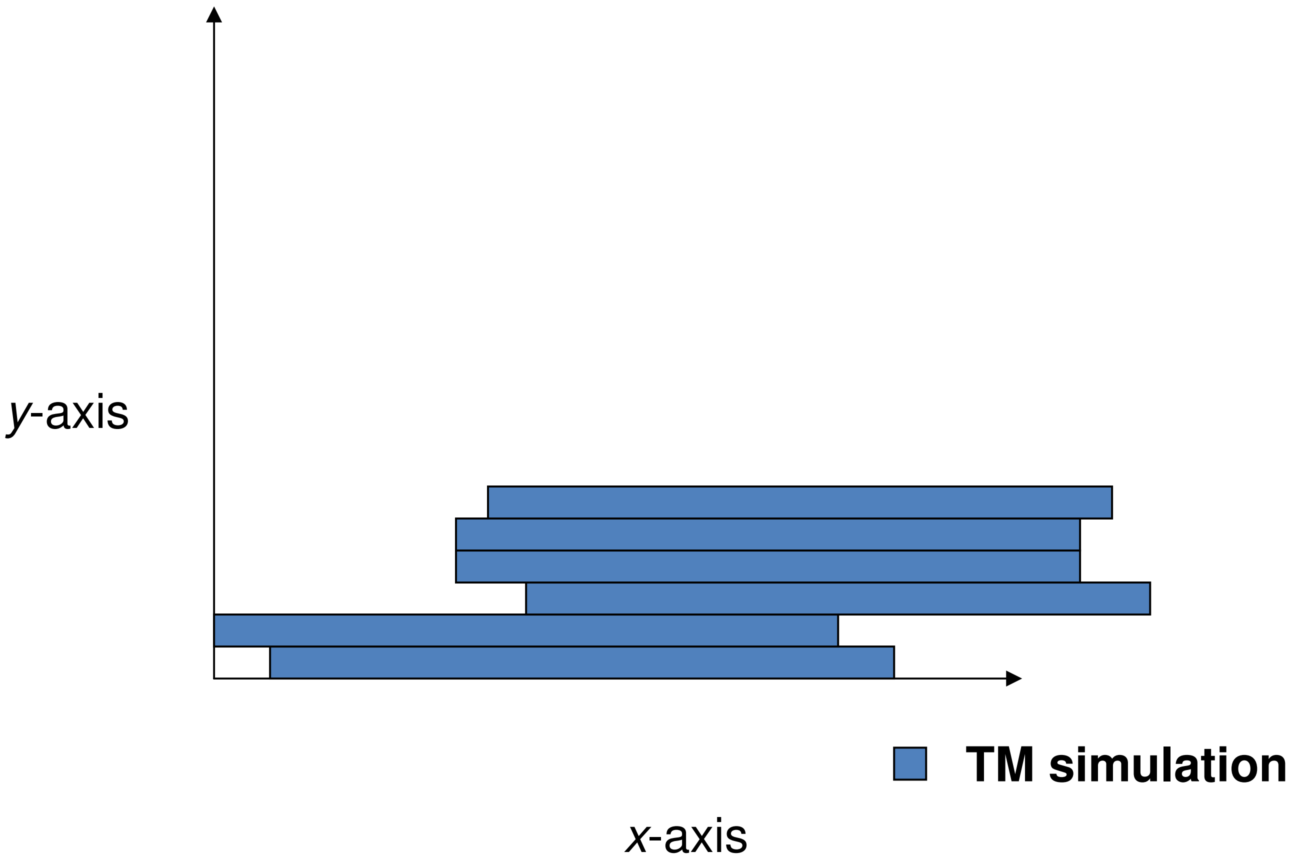

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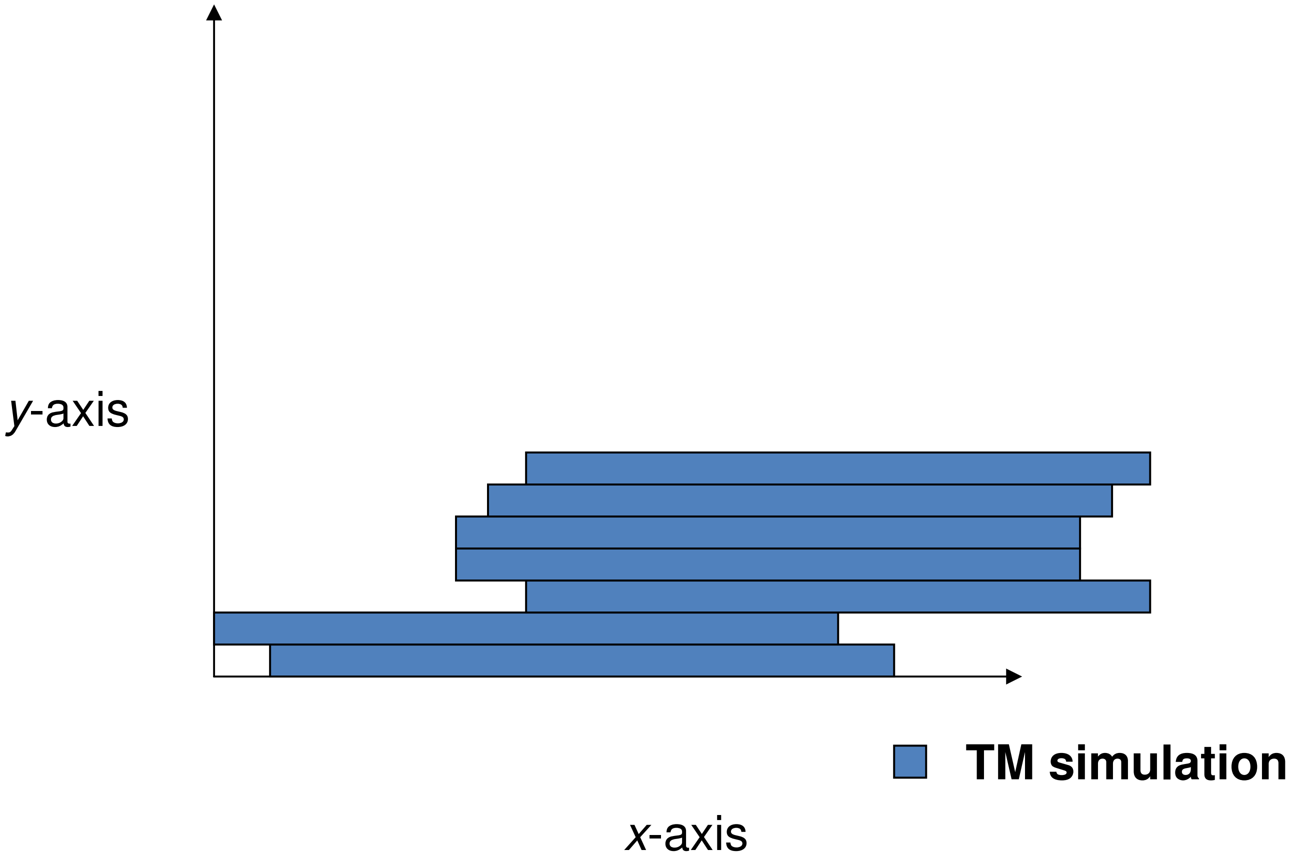

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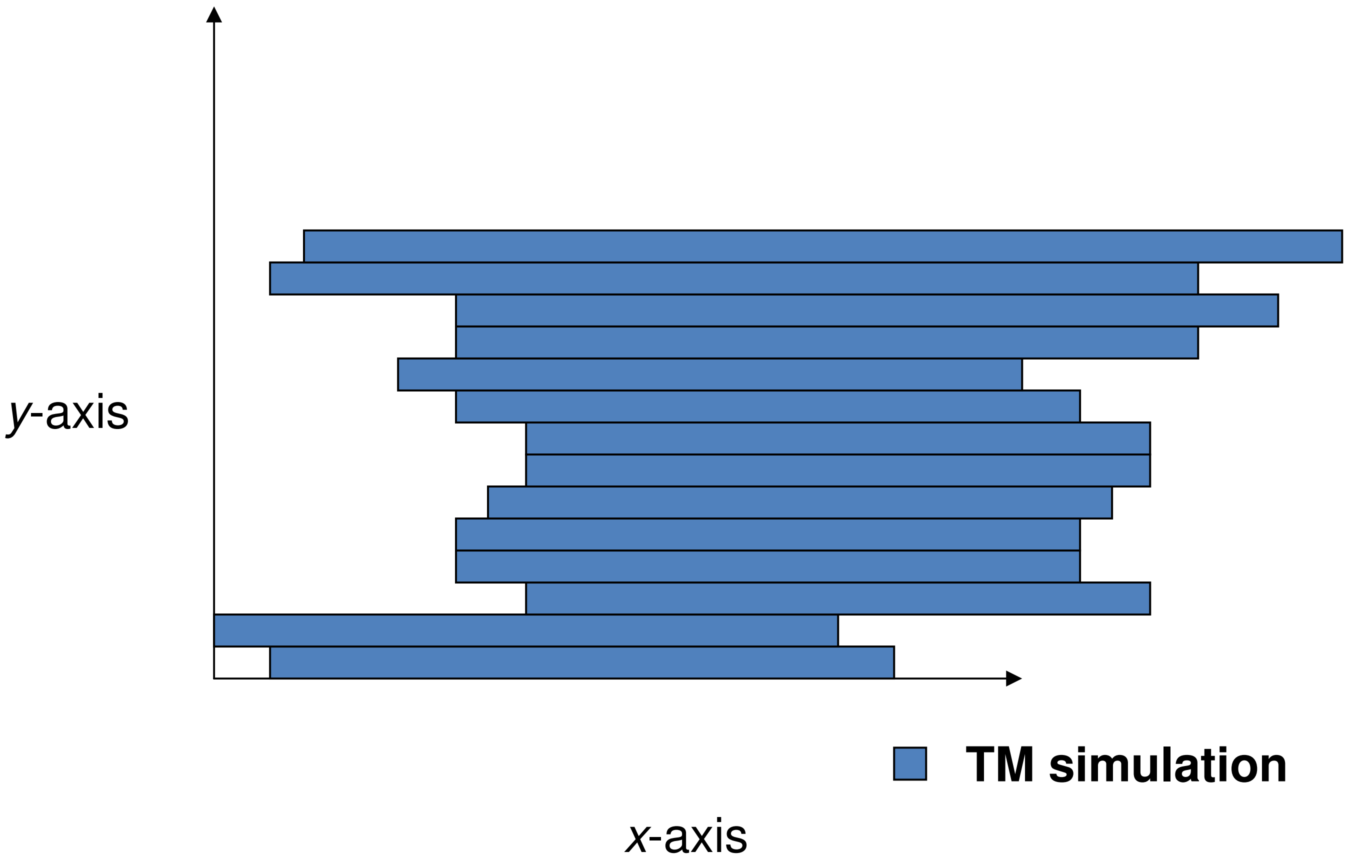

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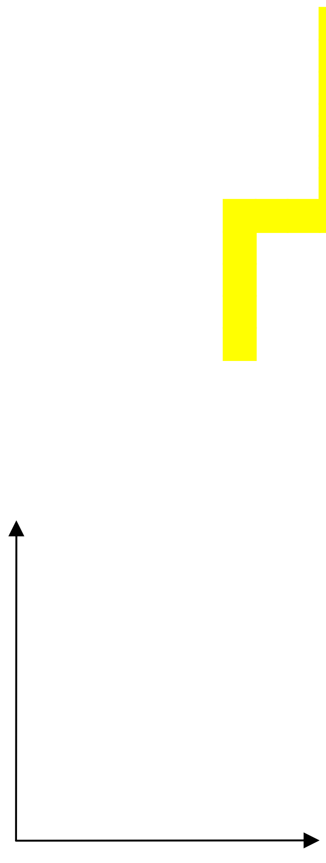
- At least one of the yellow or black paths must turn “left” at some point (otherwise the language has small space complexity)
- Once a single path turns left, all must do so.
- Infinitely many paths turning left must eventually creep into another (adjacent) quadrant.

Second Main Result (proof idea)



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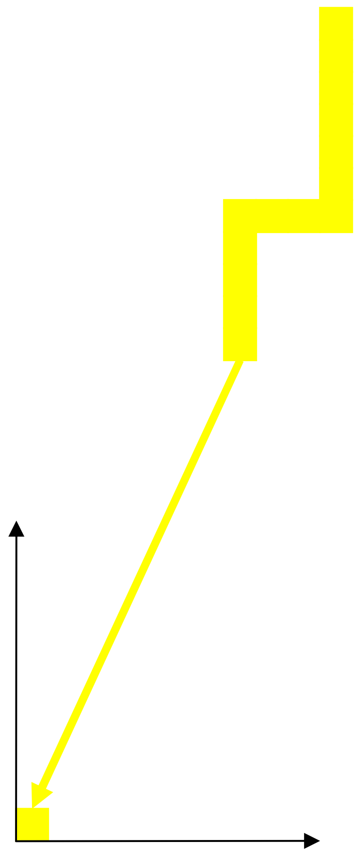

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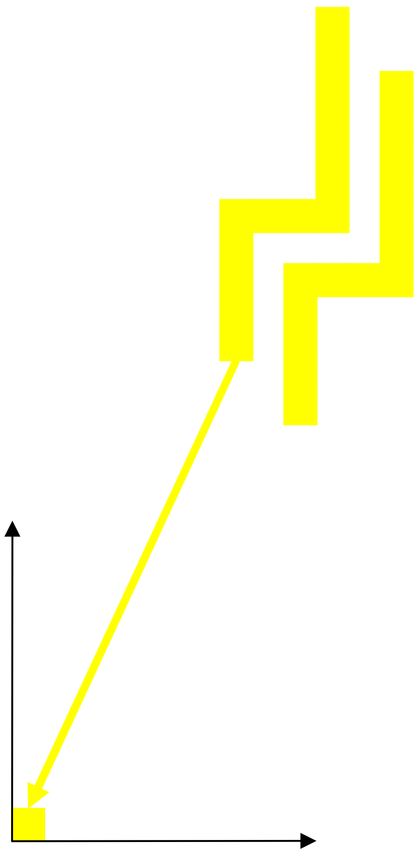

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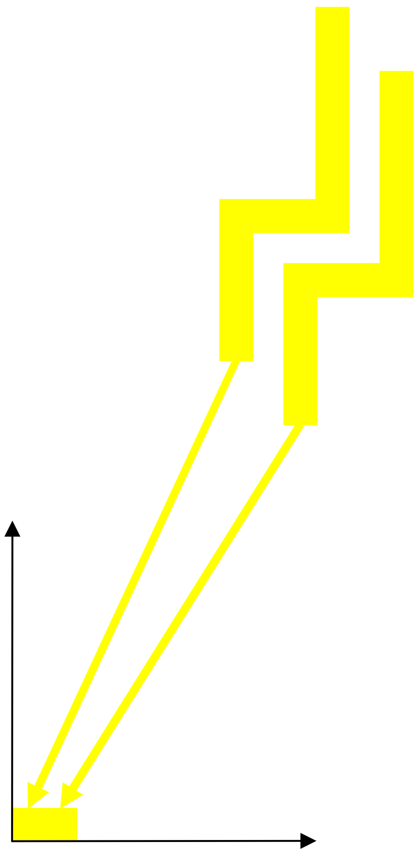

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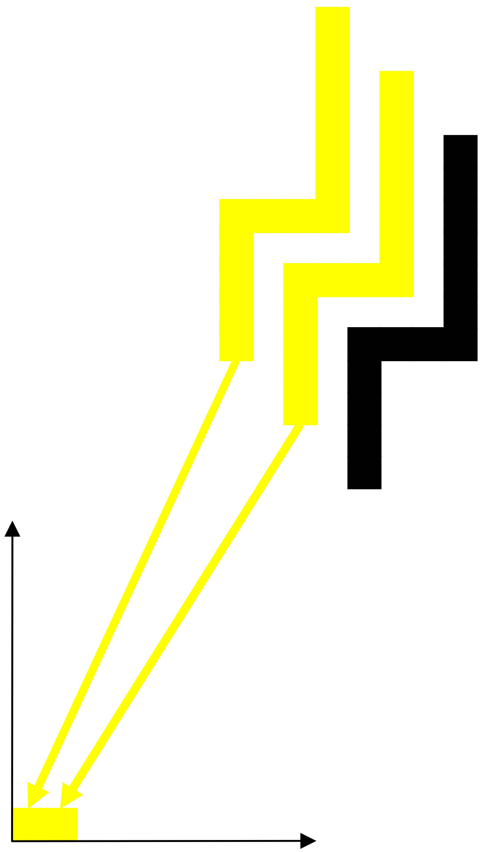

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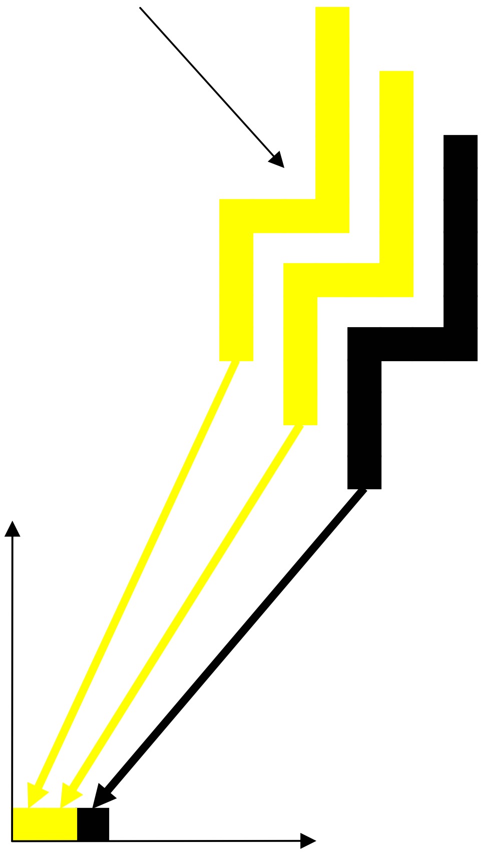

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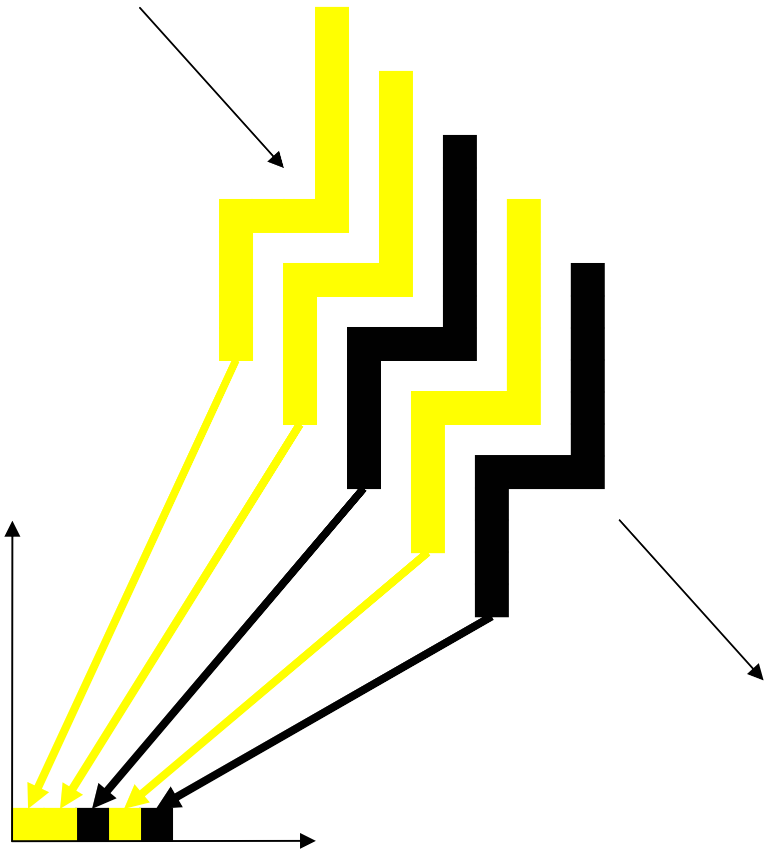

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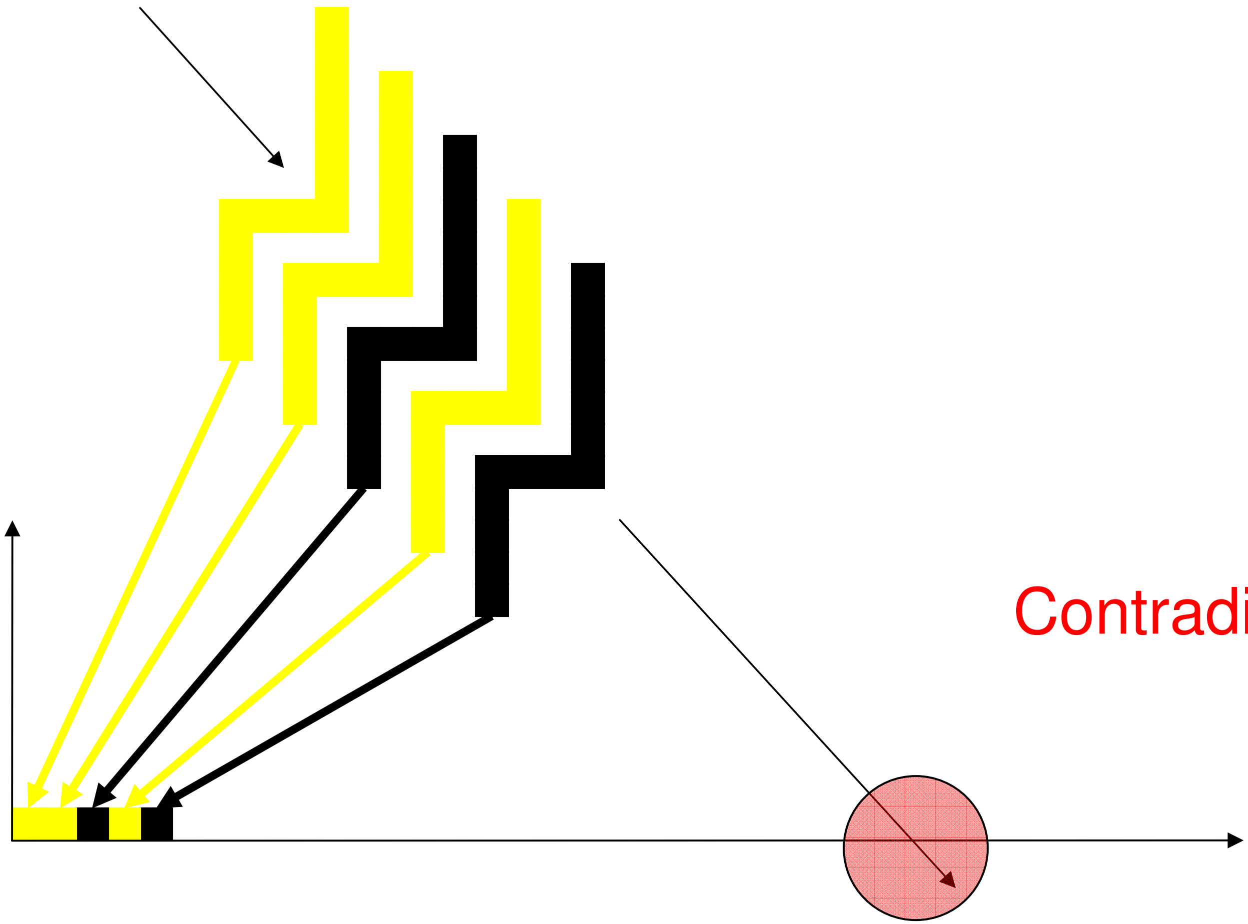

Contradiction!

Web Site

Software which generates tile sets for the main construction from Turing machine definitions and tile assembly simulation software are freely available for download from the homepage for the ISU Laboratory for Nanoscale Self-Assembly:

http://www.cs.iastate.edu/~lnsa

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Thank you!