Computational bounds on polynomial differential equations

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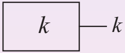
What can be achieved with analog computation?

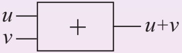
- **Church-Turing Thesis:** Every function *computable according to the intuitive notion of algorithm* is computable by a Turing machine
- But what can we say about analog computational models that do not rely on discrete procedures?

The GPAC

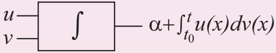
In 1941, Claude Shannon presented a paper entitled "Mathematical theory of the Differential Analyzer", where he first described the *General Purpose Analog Computer* (GPAC).



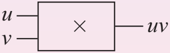
A constant unit associated to the real value k



An adder unit



An integrator unit

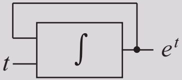


A multiplier unit

Example

Example

Compute $y(x) = e^x$ with a GPAC



$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

Main features of the GPAC

- Real numbers are not treated as strings of digits
- Assumes continuous-time dynamics
- The computation is performed in *real time*: for a GPAC computing a function f, if an input x is given at time t, the output at time t is $f(x)$, i.e. the computation took 0 time units to be carried out.
- Generates analytic functions (i.e. it has smooth dynamics)

Proposition

A scalar function $f : \mathbb{R} \to \mathbb{R}$ is generated by a GPAC iff it is PIVP function i.e. a component of the solution of a system

$$y' = p(t, y),$$

where p is a vector of polynomials.

Questions

1. Can PIVP functions (i.e. a GPAC) simulate Turing machines?
2. Can the previous simulation be made robust?
3. Which are the “simplest” PIVP functions to be able to simulate Turing machines?
4. Can we establish limits on what can be computed (simulated) by PIVP functions?

Properties of PIVP functions

The PIVP functions are closed under the following operations (as far as we know, these properties have only been reported in the literature for the broader case of differentially algebraic functions):

- Field operations $+, -, \times, /$
- Composition
- Differentiation
- Compositional inverses

Corollary

All closed-form functions (i.e. elementary functions in Analysis which, informally, correspond to the functions obtained from the rational functions, sin, cos, exp *through finitely many compositions and inversions) are PIVP functions.*

Theorem

Let S be a subfield of $\mathbb{R}$. Consider the IVP

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where $f : D \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^n$, D is the domain of f, and each component of f is a composition of polynomials with coefficients in S and PIVP functions with parameters in S and $(t_0, x_0) \in D \cap S^{n+1}$. Then there exists $m \geq n$, a polynomial $p : \mathbb{R}^{m+1} \to \mathbb{R}^m$ with coefficients in S and $y_0 \in S^m$ such that the solution of (1) is given by the first n components of $y = (y_1, ..., y_m)$, where y is the solution of the PIVP

$$\begin{cases} y' = p(t,y) \\ y(t_0) = y_0 \end{cases}$$

Robust simulations of discrete DSs

Before trying to answer questions about the simulation of discrete dynamical systems via continuous ones, we have to define what we mean by "simulation".

Let $\mathcal{D}$ be a discrete dynamical system (time and space are discrete). Each point of the state space can be coded as a point in $\mathbb{N}^m$ so that the evolution of the system is modeled by the iteration of a map $\omega : \mathbb{N}^m \to \mathbb{N}^m$. In general, if f is a function, we denote its kth iterate by $f^{[k]}$, i.e. $f^{[0]}(x) = x$ and $f^{[k+1]} = f \circ f^{[k]}$ for all $k \in \mathbb{N}$. We now present some definitions.

Definition

The map $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ is a (real) *robust extension* of the map $\omega : \mathbb{N}^m \to \mathbb{N}^m$ if there exist $\delta_{in}, \delta_{ev}, \delta_{out} \in (0, 1/2)$ such that for all $x_0 \in \mathbb{R}^m$, $n_0 \in \mathbb{N}^m$, $\overline{\Omega} : \mathbb{R}^m \to \mathbb{R}^m$ one has

1. $\Omega(n) = \omega(n)$ and
2. $\|n_0 - x_0\|_\infty \leq \delta_{in}$ and $\left\|\Omega - \overline{\Omega}\right\|_\infty \leq \delta_{ev}$ implies $\left\|\omega(n_0) - \overline{\Omega}(x_0)\right\|_\infty \leq \delta_{out}$.

Lemma

If $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ is a robust extension of the map $\omega : \mathbb{N}^m \to \mathbb{N}^m$, then there exist $\delta_{in}, \delta_{ev}, \delta_{out} \in (0, 1/2)$ such that for all $x_0 \in \mathbb{R}^m$, $n_0 \in \mathbb{N}^m$, $\overline{\Omega} : \mathbb{R}^m \to \mathbb{R}^m$ one has

1. $\Omega(n) = \omega(n)$ *and*
2. $\|n_0 - x_0\|_\infty \leq \delta_{in}$ *and* $\left\|\Omega - \overline{\Omega}\right\|_\infty \leq \delta_{ev}$ *implies* $\left\|\omega^{[k]}(n_0) - \overline{\Omega}^{[k]}(x_0)\right\|_\infty \leq \delta_{out}$ *for all* $k \in \mathbb{N}$.

Definition

Let $\phi : \mathbb{R} \to \mathbb{R}$ be the unique solution of the initial value problem

$$x' = f(t, x), \quad x(0) = n_0.$$

We say that ϕ is a *robust suspension* of the map $\omega : \mathbb{N}^m \to \mathbb{N}^m$ if there exist δ_{in}, δ_{ev}, δ_{out}, $\delta_{time} \in (0, 1/2)$, such that for all $x_0 \in \mathbb{R}^m$, $n_0 \in \mathbb{N}^m$, $k \in \mathbb{N}$, and $\overline{f} : \mathbb{R}^{m+1} \to \mathbb{R}^m$ one has that

$$\|n_0 - x_0\|_\infty \leq \delta_{in} \text{ and } \left\|\overline{f} - f\right\|_\infty \leq \delta_{ev}$$

implies that the solution $\overline{\phi}$ of the initial-value problem

$$x' = \overline{f}(t, x), \quad x(0) = x_0$$

satisfies

$$\left\|\omega^{[k]}(n_0) - \overline{\phi}(t)\right\|_\infty \leq \delta_{out}$$

for all $t \in \mathbb{R}_0^+$ such that $|t - k| \leq \delta_{time}$.

We shall use $\mathbb{Q}[\pi]$, the standard algebraic ring extension of $\mathbb{Q}$ by adjoining the transcendent π, and which is the smallest ring containing $\mathbb{Q} \cup \{\pi\}$:

$$\mathbb{Q}[\pi] := \{a_n\pi^n + \ldots + a_1\pi + a_0 \in \mathbb{R} | a_0, \ldots, a_n \in \mathbb{Q}\}.$$

Theorem

If the map $\omega : \mathbb{N}^m \to \mathbb{N}^m$ admits a robust extension $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ whose components are compositions of polynomials and PIVP functions with parameters in $\mathbb{Q}[\pi]$, then ω admits a robust suspension ϕ which is a vector PIVP function with parameters in $\mathbb{Q}[\pi]$.

Main results

The next proposition follows from [GCB08]. There the transition of a Turing machine is coded as a map over the integers in the following manner: we code the state as an integer and, using a representation of numbers in some adequate base, we code the right part of the tape as a second integer, and the left part as a third integer. We denote that encoding by η.

Proposition

Under the encoding η, the transition function $\omega : \mathbb{N}^3 \to \mathbb{N}^3$ of a Turing machine admits a robust extension $\Omega : \mathbb{R}^3 \to \mathbb{R}^3$. Moreover Ω can be chosen to be a composition of polynomials with coefficients in $\mathbb{Q}[\pi]$ and PIVP functions with parameters in $\mathbb{Q}[\pi]$ (in particular sin, cos and arctan).

Corollary

With the above encoding, the transition function ω of a given Turing machine admits a robust suspension ϕ. Moreover ϕ is a vector PIVP function with parameters in $\mathbb{Q}[\pi]$.

Proof of the main results

We need to iterate a map with ODEs. This is done adapting ideas originally from Branicky and further developed by Campagnolo, Costa, Moore, Graça, and Buescu.

1st construction

Consider a point $b \in \mathbb{R}$ (the *target*), some $\gamma > 0$ (the *targeting error*), and time instants t_0 (*departure time*) and t_1 (*arrival time*), with $t_1 > t_0$. Then obtain an IVP (the *targeting equation*) such that its solution y satisfies

$$|y(t_1) - b| < \gamma$$

independent of the initial condition $y(t_0) \in \mathbb{R}$.

This can be done by an ODE

$$y' = c(b - y)^3 \phi(t)$$

where $\phi : \mathbb{R} \to \mathbb{R}_0^+$ is some function satisfying $\int_{t_0}^{t_1} \phi(t)dt > 0$ and $c > 0$ is any constant which is bigger than a constant c_0 depending on γ and ϕ. Note that the only requirement for the construction to hold is that c is large enough.

2nd construction

Iterate the map $\omega : \mathbb{Z}^m \to \mathbb{Z}^m$ with a smooth ODE $y' = f(t, y)$.

Let $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ be an arbitrary smooth extension of ω to $\mathbb{R}$ (not necessarily robust). The iteration of ω may be performed (results from Campagnolo, Costa, and Moore) by the initial-value problem

$$\begin{cases} z_1' = c_1(\Omega(r(z_2)) - z_1)^3\theta_j(\sin 2\pi t) \\ z_2' = c_2(r(z_1) - z_2)^3\theta_j(-\sin 2\pi t) \end{cases} \qquad \begin{cases} z_1(0) = x_0 \\ z_2(0) = x_0, \end{cases} \tag{2}$$

where $z_1(t), z_2(t) \in \mathbb{R}^m$, $\theta_j(x) = 0$ if $x \leq 0$ and $\theta_j(x) = x^j$ if $x > 0$, and $r(x)$ is a function that is a solution of an ODE and that satisfies $r(x) = i$ whenever $x \in [i - 1/4, i + 1/4]$ for all $i \in \mathbb{Z}$. Note that c_1 and c_2 depend on j and that all coefficients in (2) are in $\mathbb{Q}[\pi]$.

Removing non-analyticity

We have used the nonanalytic functions θ_j and r which are obviously not PIVP functions. We will remove these functions using the fact that ω admits a robust extension. Therefore we have to study what happens when perturbations are allowed to prove the result.

In order to solve the previous problems, we need to recall the following two functions, σ and l_2, which were introduced and studied in [GCB08].

Lemma

Let $l_2 : \mathbb{R}^2 \to \mathbb{R}$ *be given by* $l_2(x, y) = \frac{1}{\pi} \arctan(4y(x - 1/2)) + \frac{1}{2}$. *Suppose also that* $a \in \{0, 1\}$. *Then, for any* $\bar{a}, y \in \mathbb{R}$ *satisfying* $|a - \bar{a}| \leq 1/4$ *and* $y > 0$,

$$|a - l_2(\bar{a}, y)| < \frac{1}{y}.$$

Lemma

Let $\sigma(x) = x - 0.2 \sin(2\pi x)$ *and* $\varepsilon \in [0, 1/2)$. *Then there is some contracting factor* $\lambda_\varepsilon \in (0, 1)$ *such that for all* $n \in \mathbb{Z}$, $\forall \delta \in [-\varepsilon, \varepsilon]$, $|\sigma(n + \delta) - n| < \lambda_\varepsilon \delta$.

Studying the perturbed targeting equation

Because the iterating procedure relies on the basic ODE $y' = c(b - y)^3\phi(t)$, we have to study the following perturbed version of this ODE

$$z' = c(\overline{b}(t) - z)^3\phi(t) + E(t),$$

where $\left|\overline{b}(t) - b\right| \leq \rho$ and $|E(t)| \leq \delta$. This was done in [GCB08], where it is shown that

$$|z(1/2) - b| < \rho + \gamma + \frac{\delta}{2}.$$

Removing the θ_j's

We must remove the θ_j's in two places: in the function r and in the terms $\theta_j(\pm \sin 2\pi t)$.

Since in (2) we are using a robust extension $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ of $\omega : \mathbb{N}^m \to \mathbb{N}^m$, we no longer need the corrections performed by r. There may be a problem when Ω is a robust extension of ω with $\delta_{out} > 1/4$, but this can easily be overcome by applying the function σ l times to each component of Ω until one has that $\sigma^{[l]} \circ \Omega$ is a robust extension of ω with $\delta_{in}^{\sigma} \leq 1/4$, and use $\sigma^{[l]} \circ \Omega$ instead of Ω.

On the other hand we cannot use the previous technique to treat the terms $\theta_j(\pm \sin 2\pi t)$. We need to substitute $\phi(t) = \theta_j(\sin 2\pi t)$ with an analytic (PIVP) function $\zeta : \mathbb{R} \to \mathbb{R}$ with the following ideal behavior:

(i) ζ is periodic with period 1;
(ii) $\zeta(t) = 0$ for $t \in [1/2, 1]$;
(iii) $\zeta(t) \geq 0$ for $t \in [0, 1/2]$ and $\int_0^{1/2} \zeta(t)dt > 0$.

Of course, conditions (ii) and (iii) are incompatible for analytic functions. Instead, we approximate ζ using a function ζ_ϵ, where $\epsilon > 0$. This function must satisfy the following conditions:

$(ii)'$ $|\zeta_\epsilon(t)| \leq \epsilon$ for $t \in [1/2, 1]$;
$(iii)'$ $\zeta_\epsilon(t) \geq 0$ for $t \in [0, 1/2]$ and $\int_0^{1/2} \zeta_\epsilon(t)dt > I > 0$, where I is independent of ϵ.

In [GCB08] an example of a PIVP function satisfying both $(ii)'$ and $(iii)'$ is constructed.

Application – Undecidability for PIVPs

It is well known from the basic existence-uniqueness theory of ODEs that if f is analytic, then the IVP

$$x' = f(t,x), \quad x(t_0) = x_0 \tag{3}$$

has a unique solution $x(t)$ defined on a maximal interval of existence $I = (\alpha, \beta) \subset \mathbb{R}$ that is analytic on I. The interval is maximal in the sense that either $\alpha = -\infty$ or $x(t)$ is unbounded as $t \to \alpha^+$ with similar conditions applying to β. Actually, f only needs to be continuous and locally Lipschitz in the second argument for this maximal interval to exist.

Question

Is it possible to design an automated method that, on input (f, t_0, x_0), gives as output the maximal interval of existence for the solution of (3)?

In [GZB07] it was shown that given an analytic IVP, defined with computable data, its corresponding maximal interval may be non-computable.

Non-computability results related to initial-value problems of differential equations are not new.

- (Aberth) There is an ordinary differential equation, defined with computable data, that does not have a computable solution
- (Pour-El, Richards, Zhong) There is a three-dimensional wave equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

$$\begin{aligned} u(x,y,z) &= f(x,y,z) \\ \frac{\partial u}{\partial t}(x,y,z,0) &= 0 \end{aligned}$$

  where f is computable, such that its only solution is not computable

These examples have somehow disconcerted mathematicians and physicists, since many of them accept an “extended” Church-Turing thesis: no feasible physical device has more computational power than a Turing machine

Actually, the previous results seem to rely on “ill-behaved” data:

- The first results is false if we require the solution of the ODE to be unique
- The second result is false if the solution is required to be of class C^1: if the initial velocity is C^k-computable and the initial position is C^{k-1}-computable, then the solution is C^{k-1}-computable

But this is not the case for the maximal interval problem!

Motivated by the non-computability result obtained in [GZB07], this latter paper also addresses the following problem: while it is not possible to compute the maximal interval of

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

is it possible to compute some partial information about it? In particular, is it possible to decide if this maximal interval is bounded or not?

This question has interest on its own for the following reason. In many problems, we implicitly assume that t is defined for “all time”. For example, if one wants to compute sinks or limit cycles associated with ODEs, this only makes sense if the solution of the ODE is defined for all times $t > t_0$. This is also implicitly assumed in problems like reachability. For this reason, those problems only make sense when associated with ODEs for which the maximal interval is unbounded.

In [GZB07], it was shown that for the general class of analytic IVPs, the boundedness problem of the maximal interval is undecidable. Here we will deepen this result: we will show that the boundedness problem is still undecidable for PIVPs of degree greater or equal than 56 with parameters in $\mathbb{Q}[\pi]$. Our result is slightly different in form from the case of the general class of analytic IVPs. Indeed, the coefficients of the polynomials are coded as finite sequences of integers.

The boundedness problem is decidable for linear differential equations thus implying that the boundary between decidability/undecidability lies in the class of polynomials of degree n, for some $2 \leq n \leq 56$.

This result is shown using methods which differ from those employed in [GZB07]. This result was already stated in [GBC07], but we now present its proof.

Analytic case

We now introduce a definition that allows us to compare real numbers of some given set, to avoid trivial undecidability of the boundedness problem due to the fact that the problem of deciding equality of real numbers is, in general, undecidable.

Definition

We say that a set $D \subseteq \mathbb{R}$ is effectively comparable if D has a naming system γ, if all elements of D are γ-computable, and if given γ-names of $x, y \in D$, then $x = y$ and $x < y$ are decidable

In the previous definition, "naming system" is either a (finite) notation or a (infinite) representation of the elements of D according to Weihrauch [Wei00].

Lemma

$\mathbb{Q}[\pi]$ *is effectively comparable.*

Main result

Theorem

Let D be an effectively comparable set such that $\mathbb{Q}[\pi] \subseteq D$. The following problem is undecidable: "Given $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ with polynomial components with coefficients in D (these coefficients are given by their names, as described in Definition 13), and $(t_0, x_0) \in \mathbb{Q} \times \mathbb{Q}^n$, decide whether the maximal interval of the IVP

$$\begin{cases} y' = p(t, y) \\ y(t_0) = y_0 \end{cases}$$

is bounded or not".

Actually, if we are given the description of a universal Turing machine, we can constructively define a set of polynomial ODEs simulating it that encodes the Halting Problem. If we use the small universal Turing machine presented in [Rog96], having 4 states and 6 symbols, we obtain the following theorem.

Theorem

Let D be an effectively comparable set such that $\mathbb{Q}[\pi] \subseteq D$. There is a vector $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$, with $n \geq 1$, defined by polynomials with coefficients in D (these coefficients are given by their names, as described in Definition 13), where each component has degree less than or equal to 56, such that the following problem is undecidable: "Given $(t_0, x_0) \in \mathbb{Q} \times \mathbb{Q}^n$, decide whether the maximal interval of the IVP

$$\begin{cases} y' = p(t, y) \\ y(t_0) = y_0 \end{cases}$$

is bounded or not".

Proof

The idea to prove this theorem is to simulate with a set of polynomial ODEs Rogozhin's small universal Turing machine [Rog96]. We can obtain a set of PIVPs simulating this Turing machine as described by previous results. Then we expand this PIVP system as a polynomial ODE using techniques already introduced before in this presentation. Since the entire procedure is constructive and bottom-up, it is possible to determine the degrees of the polynomials appearing in the IVP. This can be done through a careful analysis involving transcendence degrees (too many details — omitted here).

The important point is that we can obtain a PIVP function x_q, that satisfies for every $k \in \mathbb{N}$

$$\begin{cases} x_q(t) \leq m - \frac{11}{16} & \text{if } M \text{ has not halted at step } k \text{ and } t \leq k \\ x_q(t) \geq m - \frac{5}{16} & \text{if } M \text{ has already halted at step } k \text{ and } t \geq k \end{cases}$$

where the states of the Turing machine are encoded by numbers in $\{1, \ldots, m\}$ and $m = 4$ is the Halting state. Consider the IVP

$$\begin{cases} z_1' = x_q - (m - 1/2) \\ z_2 = \frac{1}{z_1} \end{cases} \iff \begin{cases} z_1' = x_q - (m - 1/2) \\ z_2' = ((m - 1/2) - x_q) z_2^2 \end{cases} \tag{4}$$

where $z_1(0) = z_2(0) = -1$. It is easy to see that while M hasn't halted, $x_q - (m - 1/2) \leq -3/16$. Thus z_1 keeps decreasing and the IVP is defined in $(0, +\infty)$, i.e. the maximal interval is unbounded, if M never halts.

On the other hand, if M eventually halts, z_1 starts increasing at a rate of at least $3/16$ and will do that forever. So, at some time it will have to assume the value 0. When this happens, a singularity appears for z_2 and the maximal interval is therefore (right-)bounded. For negative values of t just replace t by $(-t)$ in the ODE and assume t to be positive. It can be shown that the behavior of the system will be similar, and we reach the same conclusions for the left bound of the maximal interval. So M halts iff the maximal interval of the PIVP (4) is bounded, i.e. boundedness is undecidable.

Let us remark that, while the boundedness problem of the maximal interval for unrestricted PIVPs is in general undecidable, this is not the case for some subclasses of polynomials. For instance, the boundedness problem is decidable for the class of linear differential equations (the maximal interval is always $\mathbb{R}$) or for the class of one-dimensional autonomous differential equations where f is a polynomial of any degree (the ODE is separable, yielding an integral of a rational function that can be algorithmically solved). It would be interesting to investigate maximal classes where the boundedness problem is decidable.

Conclusions/perspectives

- We have shown how polynomial ODEs can simulate Turing machines
- We have shown that "well-behaved" ODEs, i. e., defined with analytic (PIVP) functions, can have non-computable properties
- It would be interesting to know which is the least degree of a polynomial ODE able to (robustly) simulate Turing machines
- It would also be interesting to know which is the least degree for which the boundedness problem for polynomial ODE is undecidable

Selected references

See the paper for a more complete listing of references.

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Thank you!