THE USE OF SCHMIDT DECOMPOSITION FOR IMPLEMENTING QUANTUM GATES

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1. INTRODUCTION

For using matrix representations of quantum gates the qubits $|0\rangle$ and $|1\rangle$ are described by the following column vectors

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad . \tag{1}$$

The transformations operating on these single qubit column vectors can be given by multiplying them by unitary matrices of dimension 2×2:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \tag{2}$$

where I is the two-dimensional unit matrix, and σ_1 , σ_2 , σ_3 are the Pauli spin matrices.

The two-qubit state can be given by four dimensional column vectors

$$\begin{aligned} |00\rangle &\equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad |01\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ |10\rangle &\equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad ; \quad |11\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \quad , \tag{3}$$

where in the ket states on the left handside of these equations the first and second number denote the state of the first and second qubit, respectively. The sign $\otimes$ represents tensor product where the two-qubit states can be described by tensor products of the first and second qubit column vectors. An input two-qubit state can be written as

$$|\psi\rangle_{in} = \left\{ \alpha|0\rangle_A|0\rangle_B + \beta|0\rangle_A|1\rangle_B + \gamma|1\rangle_A|0\rangle_B + \delta|1\rangle_A|1\rangle_B \right\} \quad , \qquad (4)$$

where the subscripts A and B refer to two qubits. $|\psi\rangle_{in}$ is given by a superposition of two-qubit states with corresponding amplitudes α,β,γ and δ . The CNOT gate is defined as leading to the output state

$$|\psi\rangle_{out} = \left\{ \alpha|0\rangle_A|0\rangle_B + \beta|0\rangle_A|1\rangle_B + \gamma|1\rangle_A|1\rangle_B + \delta|1\rangle_A|0\rangle_B \right\} \quad . \qquad (5)$$

The first bit (A) acts as a control and its value is unchanged on the output. The second (target) bit (B) is flipped ($|0\rangle \to |1\rangle\,;|1\rangle \to |0\rangle$) if and only if the first bit is set to one. Such gate is quite difficult to implement since the state of the control qubit should affect the second target qubit and this requires strong interactions between single photons. Such interactions need high nonlinearities well beyond what is available experimentally. Similar problems arise for other two-qubit gates.

Quite often one starts with a separable two-qubit state which is written as

$$|\psi\rangle_{in} = \left(\alpha|0\rangle_A + \beta|1\rangle_A \right)\left(\gamma|0\rangle_B + \delta|1\rangle_B \right) \qquad (6)$$

where, α, β and γ, δ are the amplitudes of the first and second qubit ,respectively, and then the $CNOT$ gate produces the *entangled* state given by

$$|\psi\rangle_{out} = \{\alpha\gamma|0\rangle_A|0\rangle_B + \alpha\delta|0\rangle_A|1\rangle_B + \beta\gamma|1\rangle_A|1\rangle_B + \beta\delta|1\rangle_A|0\rangle_B\}. \quad (7)$$

The $CNOT$ gate transformation of a separable state from (6) to (7) is a special case, of the more general CNOT transformation from (4) to (5) in which the initial state could already be entangled.

The operation of $CNOT$ on the four dimensional vectors is given by the matrix

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \quad (8)$$

There are other two-qubit gates and we are treating especially

$$CPHASE(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix} . \quad (9)$$

This gate which does not have a classical analog [see e.g. M.A. Nielsen and Chuang I.L. "Quantum Computation etc."]) applies a phase shift for the state $|1,1\rangle$ giving

$$CPHASE(\theta)|1,1\rangle = e^{i\theta}|1,1\rangle, \quad (10)$$

and does nothing if it operates on other states.

The present work study the possibility to implement these gates by the Hilbert-Schmidt decomposition.

2.THE HILBERT SCHMIDT DECOMPOSITION OF QUANTUM GATES

Any two-qubit gate operating on the two qubit states can be given by multiplying the four dimensional vectors of (3) by a unitary matrix U_2 of dimension 4×4. Any such matrix can be represented in the Hilbert Schmidt decomposition as

$$U_2 = \sum_{l,m=0}^{3} t_{l.m}\sigma_l \otimes \sigma_m \quad . \tag{11}$$

The $CNOT$ gate is decomposed as

$$CNOT = \frac{1}{2}\{\sigma_3 \otimes I - \sigma_3 \otimes \sigma_1 + I \otimes \sigma_1 + I \otimes I\} \tag{12}$$

For the $CPHASE(\theta)$ gate of (9) we get:

$$CPHASE(\theta) = \kappa(I \otimes I) + \lambda(\sigma_3 \otimes \sigma_3) + \mu(I \otimes \sigma_3) + \nu(\sigma_3 \otimes I) \tag{13}$$

where

$$\lambda = \frac{1}{4}\left(e^{i\theta} - 1\right) \quad , \quad \mu = \nu = -\lambda \quad , \quad \kappa = 1 + \lambda \tag{14}$$

For the very important special case for which $\theta = \pi$ the $CPHASE(\pi)$ parameters are reduced to

$$\lambda = -\frac{1}{2} \quad , \quad \mu = \nu = \frac{1}{2} \quad , \quad \kappa = \frac{1}{2} \ . \tag{15}$$

We find that in the decomposition (12) we have four unitary matrices multiplied by 1/2. In (13) we have four unitary matrices multiplied by different constants .In (15) we have four unitary matrices multiplied by 1/2 with different signs.

In polarization optics the states represented by the column vectors of (1) are known as Jones vectors. By using Jones calculus it is quite easy to implement the unitary transformations operating on single qubit column vectors.(See e.g. polarization optics transformations obtained by

Jones calculus in A.Yariv "optical electronics"), One should notice that each of the unitary Pauli matrices can operate only on one qubit given as

$$\begin{aligned} &\sigma_3|0\rangle=|0\rangle \quad ; \quad \sigma_3|1\rangle\equiv-|1\rangle \quad ; \quad \sigma_1|0\rangle=|1\rangle \quad ; \quad \sigma_1|1\rangle=|0\rangle \quad ; \\ &I|0\rangle=|0\rangle \quad ; \quad I|1\rangle=|1\rangle \end{aligned} \tag{16}$$

We find that by using the Jones calculus it is relatively easy to implement any of the unitary matrices composing the CNOT. The problem is that in Quantum Computation we use *multiplication* of unitary matrices and not summation of such matrices.So the idea was to use *quantum encoders* which copy each qubit few times.

3. COPYING QUBITS BY QUANTUM ENCODERS

The encoder consists primarily of a polarizing beam splitter (PBS) and resource pair of entangled photons in the Bell state $|\phi^+\rangle=\frac{|00\rangle+|11\rangle}{\sqrt{2}}$. For the quantum encoder the input qubit of a single photon , in a general polarization state $\alpha|0\rangle+\beta|1\rangle$, and one member of the entangled source pair are mixed at the PBS oriented in the HV basis. There are three outputs of the quantum encoder, including two output ports for the PBS and one output port for the second member of the entangled source pair. Detection of one photon and only one (1AO1) by 'gating detector' in one output port of the PBS signals the fact that the two remaining photons are exiting the device in the other two output ports. Because the PBS transmitts H-polarized photons and reflects V-polarized photons it can be shown (see Refs. 5-7) that if we accept the remaining outputs only when the condition $1AO1$ is satisfied by the gating detector then the device realizes the encoding :

$$\alpha|0\rangle+\beta|1\rangle \to \alpha|0\rangle_1|0\rangle_2+\beta|1\rangle_1|1\rangle_2 \ . \tag{17}$$

In order to have the condition $1AO1$ and erase any additional information obtained by the gating detector the encoding is completed by accepting the gating detector outout output only when the gating detector measures exactly one photon in a polarization basis rotated by 45^0 from the HV basis . Under these circumstances and ideal conditions which occur with probability $1/2$, the device realizes the encoding (17).

The idea is to perform encoding to 4 copies as in the Figure.

We find that the input qubit $\alpha|0\rangle_{2''}+\beta|1\rangle_{2''}$ has been copied into four output qubits as

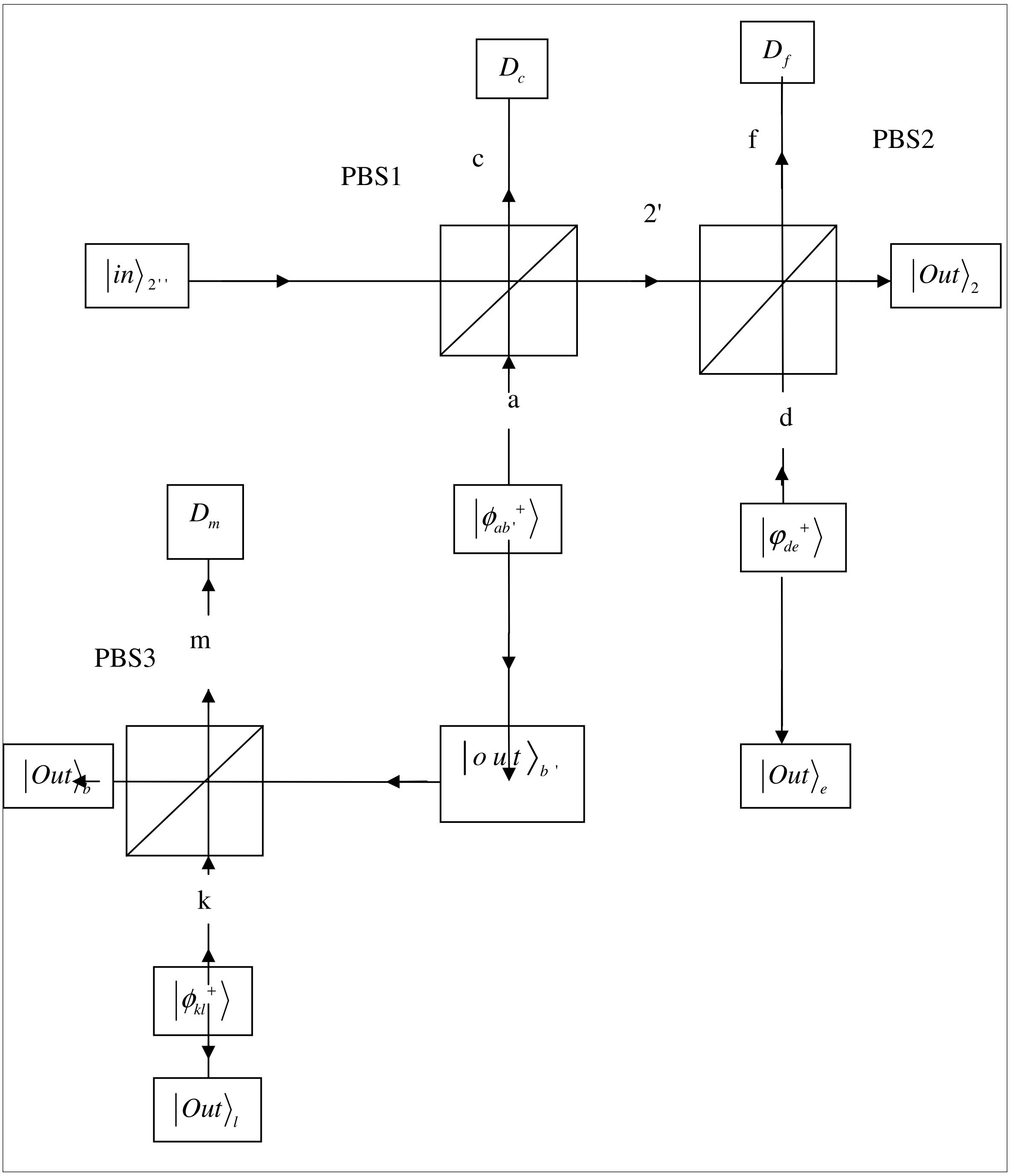
$$\alpha|0\rangle_{2''}+\beta|1\rangle_{2''} \to \alpha|0\rangle_2|0\rangle_e|0\rangle_b|0\rangle_l+\beta|1\rangle_2|1\rangle_e|1\rangle_b|1\rangle_l \quad . \tag{18}$$

Since the transformation (18) is based on the use of three PBS the total operation of this encoder succeeds with probability $\left(\frac{1}{2}\right)^3=\frac{1}{8}$.

By using a similar procedure analogous to that represented by Figure 1 , a second input qubit can be copied into four output qubits. This transformation can be represented as

$$\gamma|0\rangle_{2''}+\delta|1\rangle_{2''} \to \gamma|0\rangle_P|0\rangle_Q|0\rangle_R|0\rangle_S+\delta|1\rangle_P|1\rangle_Q|1\rangle_R|1\rangle_S \quad , \tag{19}$$

The total state for separable states is given by multiplying the states of (18) and (19).



In order to see the implementation of the *CNOT* or the *CPHASE* gate we describe the encoded state for separable states as multiplication of (18) and (19) and reorder the terms in the four multiplications as

$$\begin{aligned}
&|\psi\rangle_{inc} = \alpha\gamma\{|0\rangle_2|0\rangle_P\}\{|0\rangle_e|0\rangle_Q\}\{|0\rangle_b|0\rangle_R\}\{|0\rangle_l|0\rangle_S\}+\\
&\alpha\delta\{|0\rangle_2|1\rangle_P\}\{|0\rangle_e|1\rangle_Q\}\{|0\rangle_b|1\rangle_R\}\{|0\rangle_l|1\rangle_S\} \quad +\\
&\beta\gamma\{|1\rangle_2|0\rangle_P\}\{|1\rangle_e|0\rangle_Q\}\{|1\rangle_b|0\rangle_R\}\{|1\rangle_l|0\rangle_S\} \quad +\\
&\beta\delta\{|1\rangle_2|1\rangle_P\}\{|1\rangle_e|1\rangle_Q\}\{|1\rangle_b|1\rangle_R\}\{|1\rangle_l|1\rangle_S\},
\end{aligned} \tag{20}$$

where the two-qubit multiplications of (20) represent four copies of the two-qubit $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|1\rangle|0\rangle$, $|1\rangle|1\rangle$, respectively. Notice that the multiplications factors $\alpha\gamma$, $\alpha\delta$, $\beta\gamma$, $\beta\delta$ introduce only multiplication factors which will not change the following analysis. The production of the encoded state given by (20) enables us to realize the basic two-qubit quantum gates as shown in the next section. We restricted the copying process for the special case where the initial state is separable. More general 'entangled encoding' processes can generalize the copying process for the cases in which the initial state is already entangled.

4.MATHEMATICAL REALIZATION OF *CNOT* AND *CPHASE* GATES

By operating with the four unitary matrices of the decomposition (12) on the four copies of the $|0\rangle|0\rangle$ two-qubit (given in the first line of (20)) we get

$$\begin{aligned}
&\sigma_3 \otimes I\,|0\rangle|0\rangle = |0\rangle|0\rangle \quad , \quad -\sigma_3 \otimes \sigma_1\,|0\rangle|0\rangle = -|0\rangle|1\rangle \quad , \\
&I \otimes \sigma_1\,|0\rangle|0\rangle = |0\rangle|1\rangle \quad , \quad I \otimes I\,|0\rangle|0\rangle = |0\rangle|0\rangle
\end{aligned} \quad . \tag{21}$$

By (21) we find four output two-qubit states. If we add the amplitudes for the four two-qubits we get the *mathematical CNOT transformation* $|0\rangle|0\rangle \to 2|0\rangle|0\rangle$. (The factor *2* appears due to the factor $1/2$ of (12)) and the use of the encoding procedure).

By operating with the four unitary tensor products of *CNOT* on the four copies of $|0\rangle|1\rangle$ (given in the second line of (20)) we get

$$\begin{array}{ll} \sigma_3 \otimes I\,|0\rangle|1\rangle = |0\rangle|1\rangle \quad , & -\sigma_3 \otimes \sigma_1\,|0\rangle|1\rangle = -|0\rangle|0\rangle \quad , \\ I \otimes \sigma_1\,|0\rangle|1\rangle = |0\rangle|0\rangle \quad , & I \otimes I\,|0\rangle|1\rangle = |0\rangle|1\rangle \end{array} \quad . \tag{22}$$

By adding the amplitudes of the four two-qubit outputs we get the *mathematical CNOT transformation* $|0\rangle|1\rangle \to 2|0\rangle|1\rangle$.

By operating with the four unitary tensor products of *CNOT* on the four copies of $|1\rangle|0\rangle$ (given in the third line of (20)) we get

$$\begin{array}{ll} \sigma_3 \otimes I\ \ |1\rangle|0\rangle = -|1\rangle|0\rangle \quad , & -\sigma_3 \otimes \sigma_1\,|1\rangle|0\rangle = |1\rangle|1\rangle \quad , \\ I \otimes \sigma_1\,|1\rangle|0\rangle = |1\rangle|1\rangle \quad , & I \otimes I\,|1\rangle|0\rangle = |1\rangle|0\rangle \end{array} \quad . \tag{23}$$

By adding the amplitudes of the four two-qubit outputs we get the *mathematical CNOT transformation* $|1\rangle|0\rangle \to 2|1\rangle|1\rangle$.

By operating with the four unitary tensor products of *CNOT* on the four copies of $|1\rangle|1\rangle$ (given in the fourth line of (20)) we get

$$\begin{array}{ll} \sigma_3 \otimes I\ \ |1\rangle|1\rangle = -|1\rangle|1\rangle \quad , & -\sigma_3 \otimes \sigma_1\,|1\rangle|1\rangle = |1\rangle|0\rangle \\ I \otimes \sigma_1\,|1\rangle|1\rangle = |1\rangle|0\rangle \quad , & I \otimes I\,|1\rangle|1\rangle = |1\rangle|1\rangle \end{array} \tag{24}$$

By adding the amplitudes of the four two-qubit outputs we get the *mathematical CNOT transformation* $|1\rangle|1\rangle \to 2|1\rangle|0\rangle$. One should take into account that although in the above calculations we separated the operations on the terms in the four lines of (20) in realization of quantum computation these operations are done in parallel.

In the same way we can get the mathematical $CPHASE(\theta)$ transformations. By operating with the four unitary matrices of the decomposition (13) on the four copies of the $|0\rangle|0\rangle$ two-qubit (given in the first line of (20)) we get by taking into account

$$CPHASE(\theta)=\kappa(I\otimes I)+\lambda(\sigma_3\otimes\sigma_3)+\mu(I\otimes\sigma_3)+\nu(\sigma_3\otimes I) \quad (13)$$

$$\lambda=\frac{1}{4}\left(e^{i\theta}-1\right)\quad,\quad \mu=\nu=-\lambda\quad,\quad \kappa=1+\lambda \quad (14)$$

then

$$\begin{aligned}&\kappa(I\otimes I)\,|0\rangle|0\rangle=\kappa|0\rangle|0\rangle\quad,\quad \lambda(\sigma_3\otimes\sigma_3)\,|0\rangle|0\rangle=\lambda|0\rangle|0\rangle\quad,\\ &\mu(I\otimes\sigma_3)\,|0\rangle|0\rangle=\mu|0\rangle|0\rangle,\quad \nu(\sigma_3\otimes I)|0\rangle|0\rangle=\nu|0\rangle|0\rangle\end{aligned}\quad . \; (25)$$

By (25) we find four output two-qubit states. By adding the amplitudes we get $\kappa+\lambda+\mu+\nu=1$ and then the *mathematical* $CPHASE(\theta)$ *transformation* $|0\rangle|0\rangle\rightarrow|0\rangle|0\rangle$ is realized

By operating with the four unitary tensor products of $CPHASE(\theta)$ on the four copies of $|0\rangle|1\rangle$ (given in the second line of (20)) we get

$$\begin{aligned}&\kappa(I\otimes I)\,|0\rangle|1\rangle=\kappa|0\rangle|1\rangle\quad,\quad \lambda(\sigma_3\otimes\sigma_3)|0\rangle|1\rangle=-\lambda|0\rangle|1\rangle\quad,\\ &\mu(I\otimes\sigma_3)|0\rangle|1\rangle=-\mu|0\rangle|1\rangle\quad,\quad \nu(\sigma_3\otimes I)|0\rangle|1\rangle=\nu|0\rangle|1\rangle\end{aligned}\quad .(26)$$

By adding the amplitudes of the four two-qubit outputs we get $\kappa-\lambda-\mu+\nu=1$ and then the *mathematical* $CPHASE(\theta)$ *transformation* $|0\rangle|1\rangle\rightarrow|0\rangle|1\rangle$ is realized.

By operating with the four unitary tensor products of $CPHASE(\theta)$ on the four copies of $|1\rangle|0\rangle$ (given in the third line of (18)) we get

$$\begin{aligned} &\kappa(I\otimes I)|1\rangle|0\rangle=\kappa|1\rangle|0\rangle \quad , \quad \lambda(\sigma_3\otimes\sigma_3)|1\rangle|0\rangle=-\lambda|1\rangle|0\rangle \quad , \\ &\mu(I\otimes\sigma_3)|1\rangle|0\rangle=\mu|1\rangle|0\rangle \quad , \quad \nu(\sigma_3\otimes I)|1\rangle|0\rangle=-\nu|1\rangle|0\rangle \end{aligned} \qquad .(27)$$

By adding the amplitudes of the four two-qubit outputs we get $\kappa-\lambda+\mu-\nu=1$ and the *mathematical* $CPHASE(\theta)$ *transformation* $|1\rangle|0\rangle\rightarrow|1\rangle|0\rangle$ realized.

By operating with the four unitary tensor products of $CPHASE(\theta)$ on the four copies of $|1\rangle|1\rangle$ (given in the fourth line of (20)) we get

$$\begin{aligned} &\kappa(I\otimes I)|1\rangle|1\rangle=\kappa|1\rangle|1\rangle \quad , \quad \lambda(\sigma_3\otimes\sigma_3)|1\rangle|1\rangle=\lambda|1\rangle|1\rangle \quad , \\ &\mu(I\otimes\sigma_3)|1\rangle|1\rangle=-\mu|1\rangle|1\rangle \quad , \quad \nu(\sigma_3\otimes I)|1\rangle|0\rangle=-\nu|1\rangle|1\rangle \end{aligned} \qquad (28\ )$$

By adding the amplitudes of the four two-qubit outputs we get $\kappa+\lambda-\mu-\nu=1+4\lambda=\exp(i\theta)$ and the *mathematical* $CPHASE(\theta)$ *transformation* $|1\rangle|1\rangle\rightarrow\exp(i\theta)|1\rangle|1\rangle$ is realized. One should take into account that although in the above calculations we separated the operations on the terms in the four lines of (20) these operations are done in parallel.

The above transformations implement *mathematically* the $CPHASE(\theta)$ and the $CNOT$ gates but such implementation is not enough for two reasons: a)The above mathematical transformations lead to different quantum states and it has not been shown how to add the amplitudes of the different states. b) The $CPHASE(\theta)$ and the $CNOT$ gates should be realized before any measurement. Therefore we discuss the possibility to use *decoding procedures*.

5. DECODING PROCESSES IMPLEMENTING QUANTUM GATES

We assume that the two-qubits composing each two-qubit state have different frequencies. We show a way in preliminary experiments by which the four copies of the two-qubits will have the same initial frequency.Four equivalent two-qubits are sent into one input port of PBS. Then the transformations to the output given by

$$\begin{array}{l} 4|00\rangle_{in} \rightarrow 4|00\rangle_{out} \quad ; \quad 4|01\rangle_{in} \rightarrow 4|01\rangle_{out} \quad ; \\ 4|10\rangle_{in} \rightarrow 4|10\rangle_{out} \quad ; \quad 4|11\rangle_{in} \rightarrow 4|11\rangle_{out} \end{array} \quad , \qquad (29)$$

guarantee that the four two-qubits have the same initial phase. In (29) we use the property of the PBS that *totally* transmitts horizontal polarized photons and reflects vertical polarized photons. Such transformations in preliminary experiments can be accomplished by the use of phase shifters. Then by using the unitary transformations in the quantum quantum gate transformations (29) are changed from the initial states before the quantum encoders to final states after the decoding interference.

One should notice the following results:

a) Since the CNOT gate decomposition is given by the summation of 4 unitary matrices multiplied by the common factor 1/2 we can add the results by the above interference experiment. The transformations will be equal to the CNOT transformation up to a multiplication constant .

b) Since the $CPHASE(\pi)$ gate decomposition is given by the summation of 4 unitary matrices multiplied by the common factor 1/2 (up to a change of sign) we can implement the $CPHASE(\pi)$ gate by interference experiment up to a multiplication constant .

c) In the general case of $CPHASE(\theta)$ where $\theta \neq \pi$ it will be quite difficult to implement the decoding since the four unitary matrices are multiplied by different constants given by (15).

5. CONCLUSIONS

a) The incoding process is of much interest by itself in quantum computation and the present method can be useful for other purposes.

b) The main two qubit-gates can be decomposed into a summation of four unitary matrices where each matrix will operate on a different copy of the input two-qubits obtained by quantum encoders, and the results are *mathematically* analogous to the quantum gates transformations when we add the corresponding amplitudes.

c) For the $CNOT$ and the $CPHASE(\pi)$ gates the present method can accomplish the gate transformations up to a multiplicative constant by *decoding* interference experiments.