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Quantum experiments can test mathematical undecidability

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Unconventional Computation 2008, Vienna

Theory

Johannes Kofler

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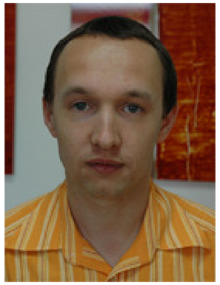


Tomasz Paterek

Experiment

Robert Prevedel

Markus Aspelmeyer

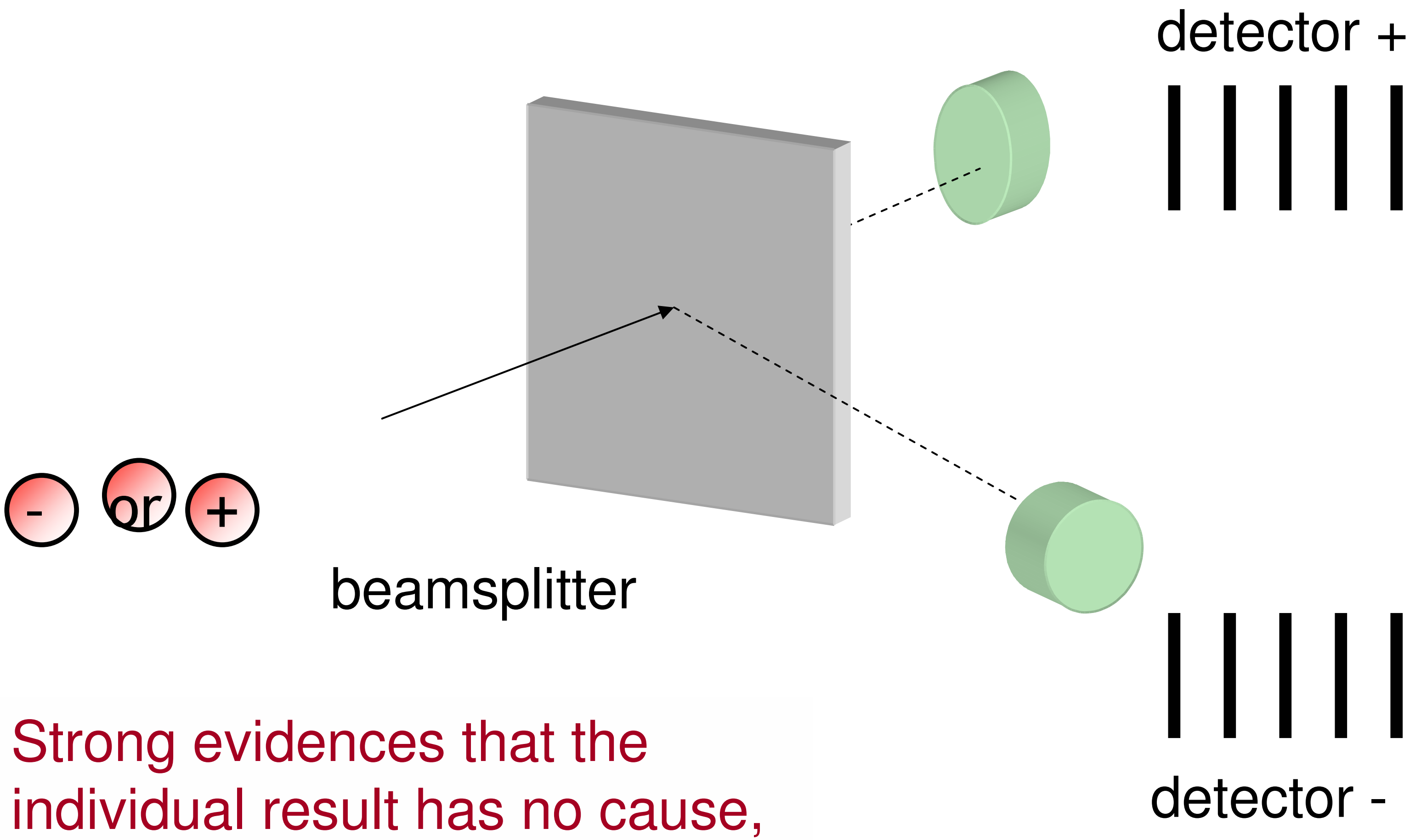
Anton Zeilinger

What is the origin of quantum randomness?

Quantum randomness (in certain experiments)
is a manifestation of mathematical undecidability.

Svozil [1990], Calude & Stay [2005] …

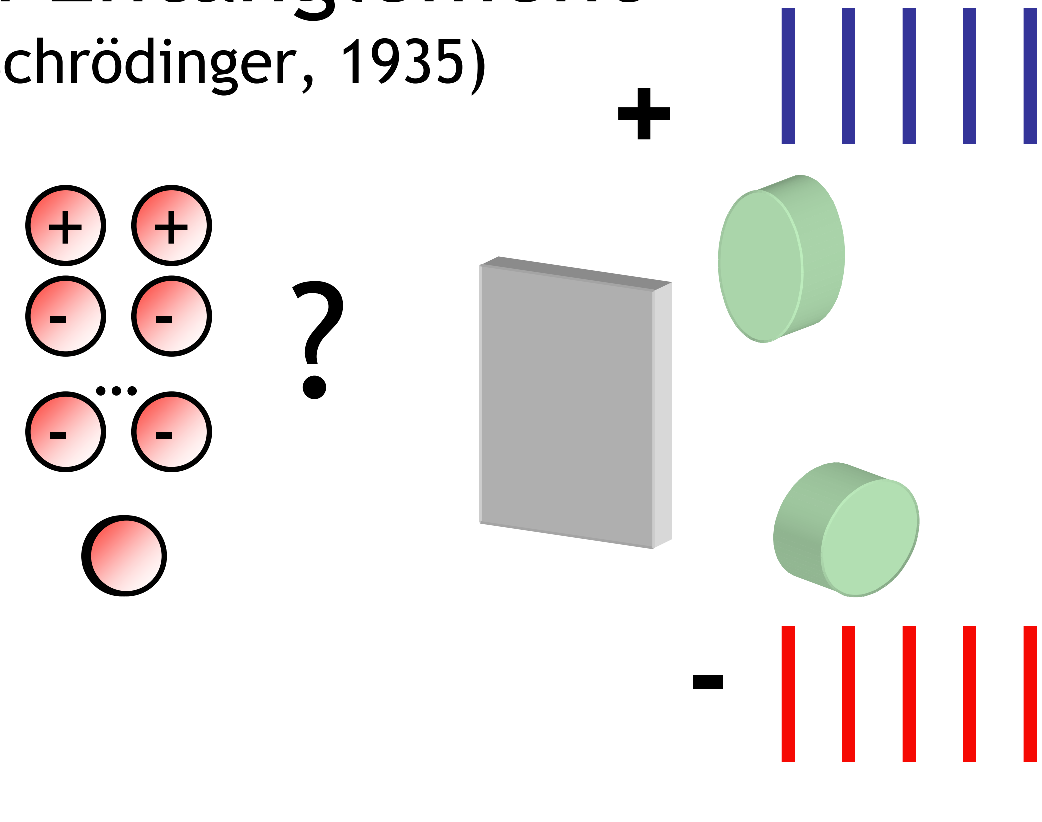
Quantum Randomness

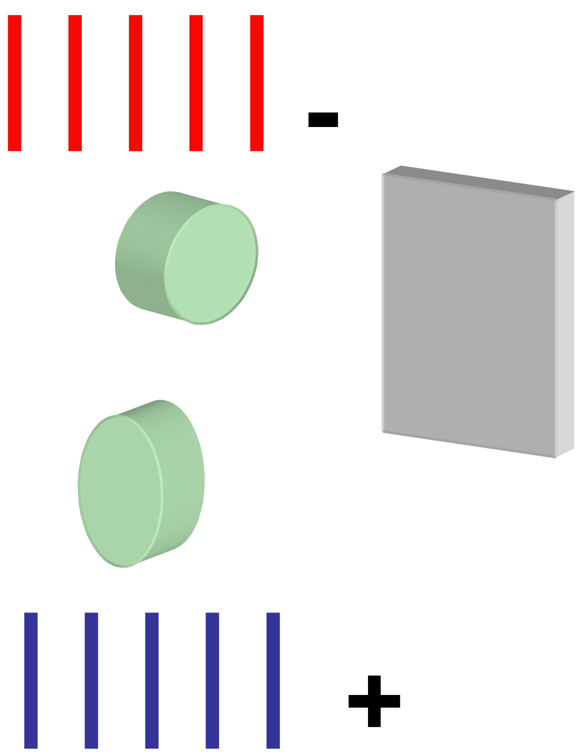


Strong evidences that the individual result has no cause, not even a hidden one

Quantum Entanglement

(Erwin Schrödinger, 1935)



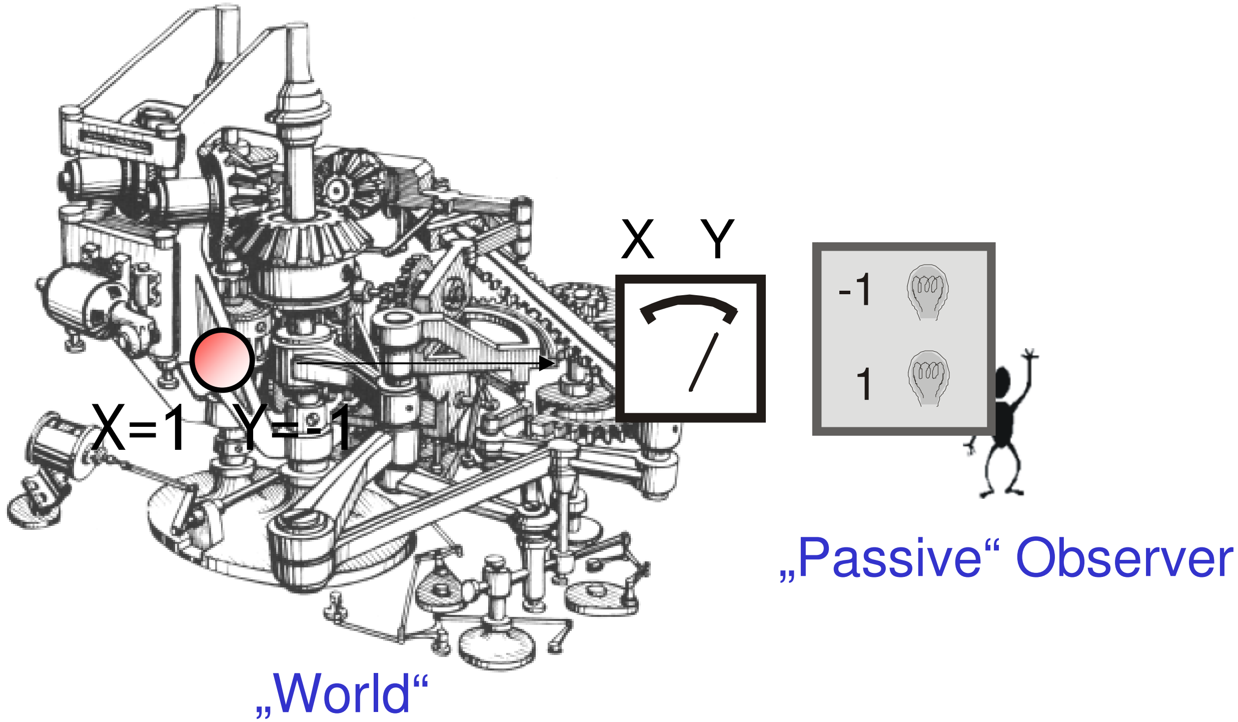


• only **joint properties** definite (perfect correlations in different measurements).

• no information on individual subsystems (local results **random**)

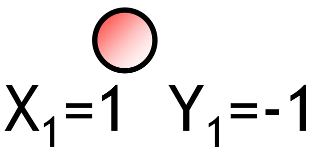
Realism:

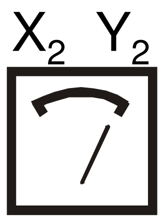
The measurement results are determined by properties the particles carry **prior to and independent** of observation.



Locality:

The results obtained at one location are **independent** of any actions performed at space-like separation.

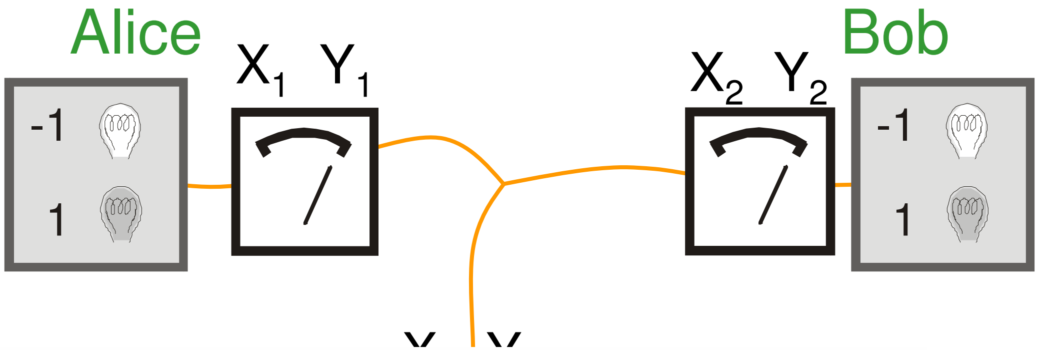




Violation of local realism

(John Bell 1964, GHZ 1989)

$$|GHZ\rangle = (|z+\rangle_1|z+\rangle_2|z+\rangle_3 + |z-\rangle_1|z-\rangle_2|z-\rangle_3)/\sqrt{2}$$



$$X_1Y_2Y_3 = -1$$

$$Y_1X_2Y_3 = -1$$

$$Y_1Y_2X_3 = -1$$

ocal Realism: $X_1X_2X_3 = -1$

Quantum: $X_1X_2X_3 = 1$

Experimental test of quantum nonlocality in three-photon Greenberger–Horne–Zeilinger entanglement

Jian-Wei Pan*, Dik Bouwmeester†, Matthew Daniell*, Harald Weinfurter‡ & Anton Zeilinger*

Nature 403, 515 (2000)

What is wrong?

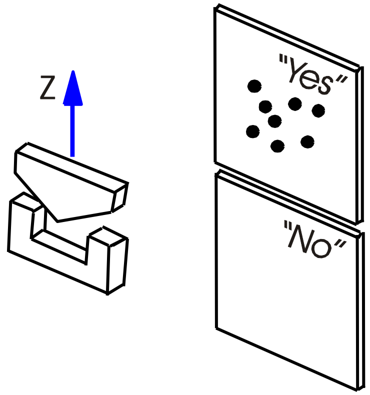
- Locality *(Tension with Special Relativity)*
- Realism

Realism requires enormous amount of informaton.

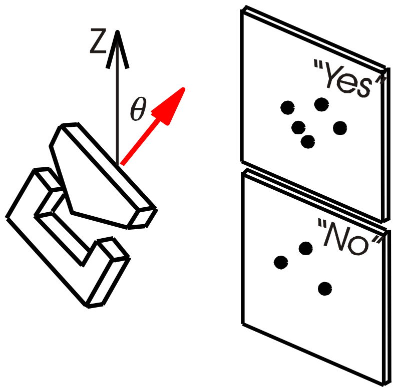
What if only a finite amount of info is available?

Finite information content

Zeilinger: The most elementary system can give a definite answer to **one** question only, i.e. it carries only 1 bit of info

- 

Proposition: *"The spin is up along z"*

- 

The answers to other questions must be random

Product & entangled states

$|\psi\rangle = |x+\rangle|x+\rangle$

1. "The spin of particle 1 is up along x".
2. "The spin of particle 2 is up along x".

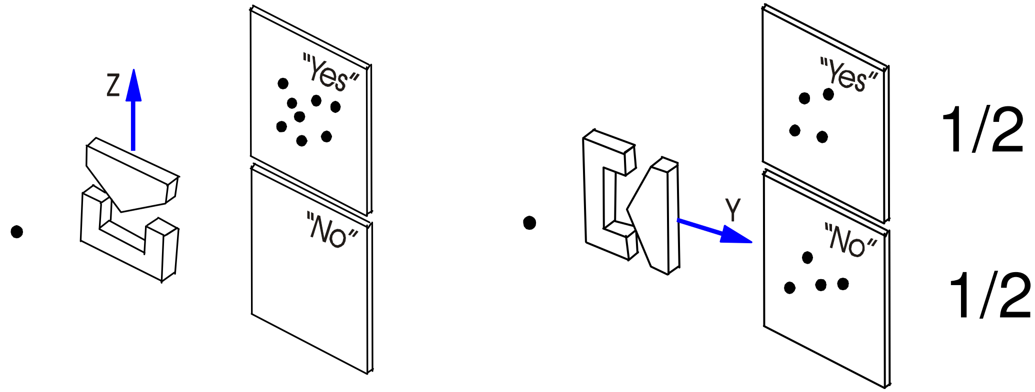
2 bits define (local) properties of individual spins.

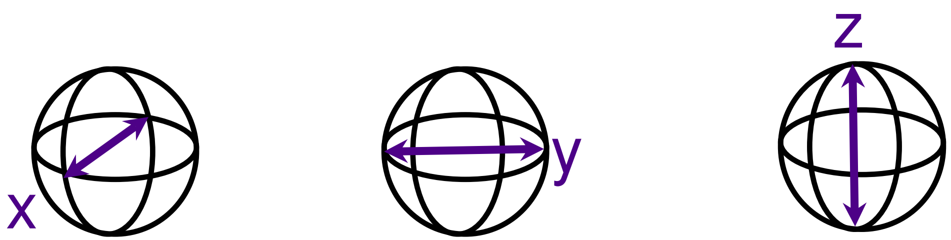
$$|\psi^{+}\rangle = \frac{1}{\sqrt{2}}\left(|x+\rangle|x-\rangle + |x-\rangle|x+\rangle\right) = \frac{1}{\sqrt{2}}\left(|y+\rangle|y+\rangle - |y-\rangle|y-\rangle\right)$$

1. "The two spins are different along x".
2. "The two spins are the same along y".

2 (nonlocal) bits define correlations. No bits left to define individual spins ⇒ **Randomness**

Mutually complementary measurements





Spin x, Spin y, Spin z

$$\left|\langle x\pm|y\pm\rangle\right|^2=\left|\langle x\pm|z\pm\rangle\right|^2=\left|\langle y\pm|z\pm\rangle\right|^2=1/2$$

Is there any relation to mathematical undecidability?

Chaitin‘s mathematical undecidability

A proposition is **undecidable** within a set of axioms,
if it can neither be proved nor disproved within the set.

Undecidability arises whenever a proposition and a given set of axioms together contain **more information** than the axioms themselves.

The one-bit axiom

Boolean functions of a binary argument

$$x \in \{0,1\} \to y = f(x) \in \{0,1\}$$

Single bit axiom: $f(0) = 0$

Proposition to be proved: $f(0) = f(1)?$

Undecidable!
Requires two bits, but the axiom contains only one.

Similarly, $f(1) = ?$ is undecidable within the axiom.

Logical complementarity

Given **limited information resources**, propositions which cannot be simultaneously ascribed definite truth values are logically complementary.

Given **1 bit** of information:

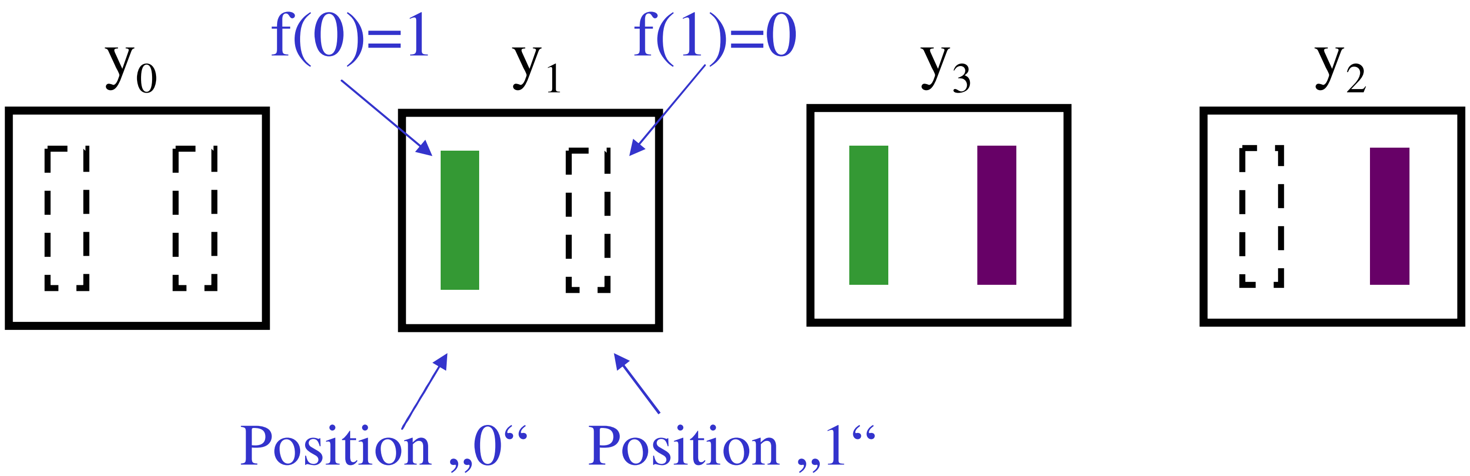
(A) $f(0) = 0$

(B) $f(1) = 0$

(C) $f(0) = f(1)$

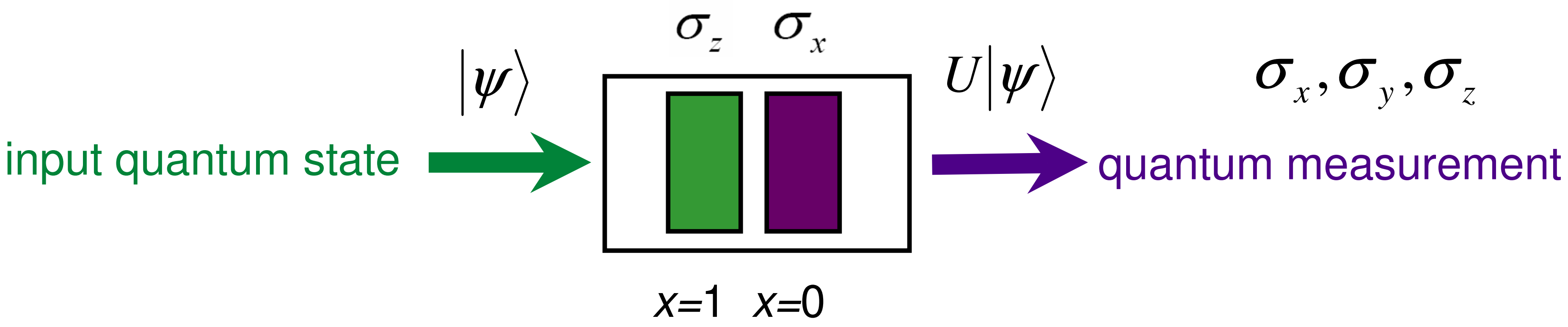
From math to physics

A black box encodes the Boolean functions
(2 bits of info)



x	y_0	y_1	y_2	y_3
0	0	0	1	1
1	0	1	0	1

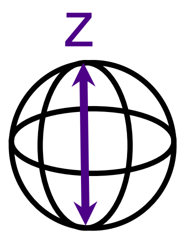
Extracting info from the box



$$U = \boldsymbol{\sigma}_x^{f(0)} \boldsymbol{\sigma}_z^{f(1)}$$

Extracting complementary bits

$|z\pm\rangle \rightarrow \boxed{U=\sigma_x^{f(0)}\sigma_z^{f(1)}} \rightarrow$

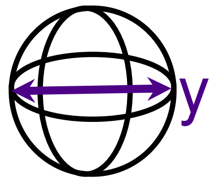


$f(0)=?$

$|x\pm\rangle \rightarrow \boxed{U=\sigma_x^{f(0)}\sigma_z^{f(1)}} \rightarrow$

$f(1)=?$

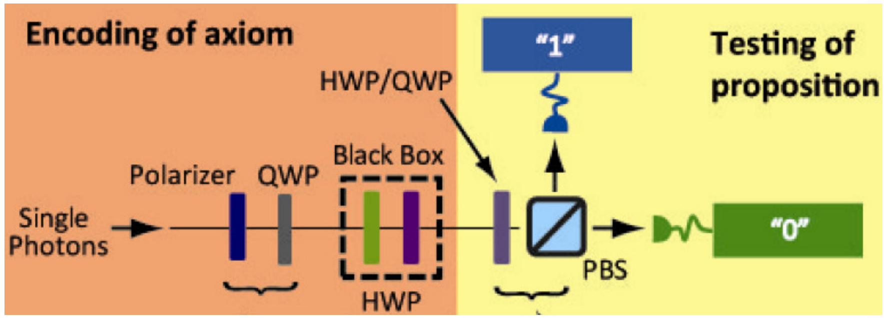
$|y\pm\rangle \rightarrow \boxed{U=\sigma_x^{f(0)}\sigma_z^{f(1)}} \rightarrow$



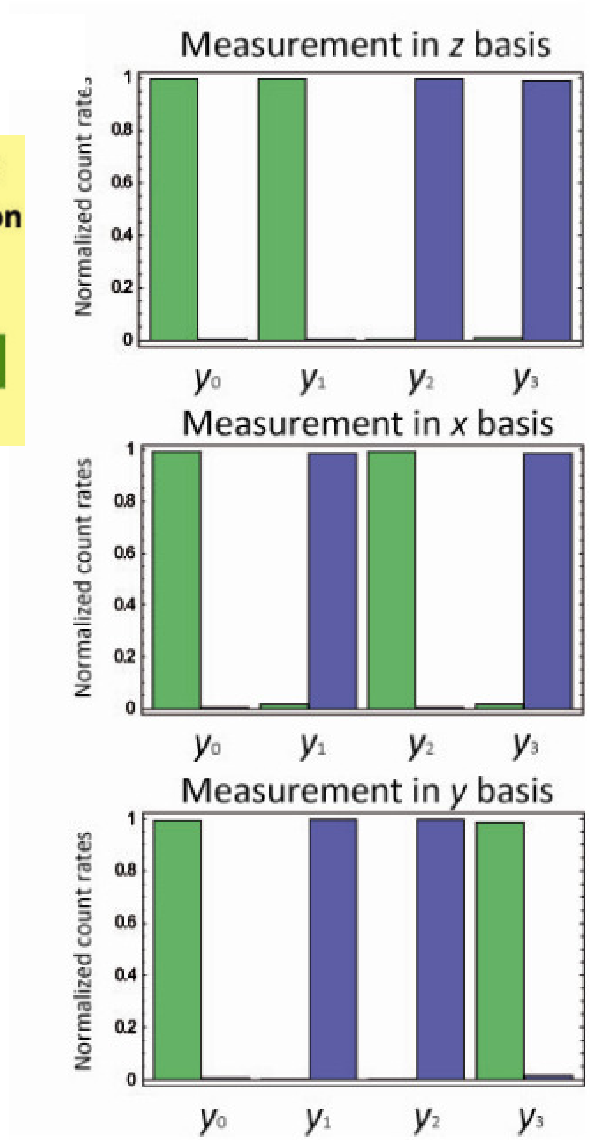
$f(0)=f(1)?$

Quantum complementary states answer logically complementary questions.

Experimental test
(photonic polarization)



x	y_0	y_1	y_2	y_3
0	0	0	1	1
1	0	1	0	1



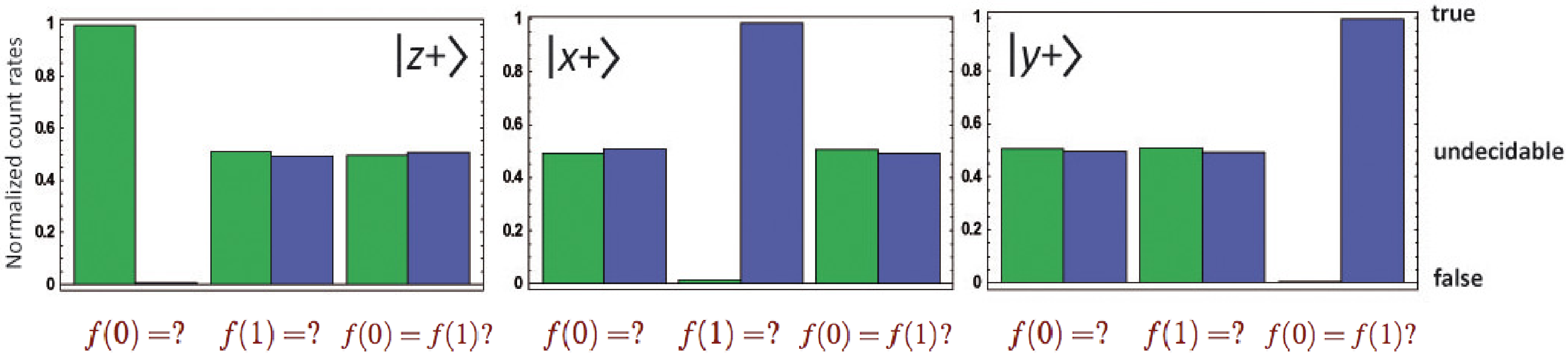
$f(0) = ?$

$f(1) = ?$

$f(0) = f(1)?$

The state encode the axiom

Randomness and undecidability

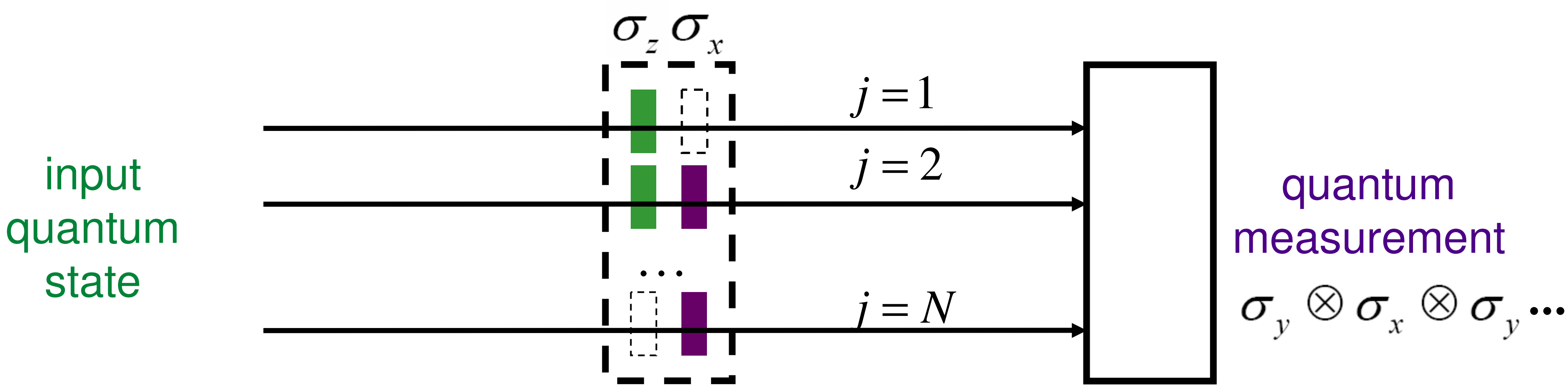


Mathematics/Logic	Quantum Physics
Axioms of limit info	State
Boolean functions	Unitary transformation
Question about proposition	Measurement
Decidability/Undecidability	Definiteness/Randomness

Whenever the proposition "$n\, f(0) + m\, f(1) = 0$" (within the 1-bit axiom system) is **decidable**, the measurement outcome is **definite**, and whenever it is **undecidable**, the outcome will be **random**.

The N-Bit Axiom

N Boolean functions $f_j(x)$ numbered by $j = 1,\ldots,N$



Black box action: $U_N = \sigma_x^{f_1(0)}\sigma_z^{f_1(1)} \otimes \cdots \otimes \sigma_x^{f_N(0)}\sigma_z^{f_N(1)}$

N Bits: Testing Propositions

Final state encode N-bit axiom

$$(\mathrm{H}_p) \quad \text{“The value of } \sum_{j=1}^{N} n_j(p)\, f_j(0) + m_j(p)\, f_j(1) = 0\text{”.}$$

$$m_j(p), n_j(p) \in \{0,1\} \quad p = 1,\ldots,N$$

Final state is an eigenstate of *N* independent and mutually commuting tensor products of Pauli operators: $\sigma_x \otimes \sigma_y \otimes \sigma_y \cdots$

Measurement is a test of proposition

$$(\mathrm{J}) \quad \text{“The value of } \sum_{j=1}^{N} \beta_j\, f_j(0) + \alpha_j\, f_j(1) = 0\text{”}$$

Measurement is a product of Pauli operators: $\sigma_y \otimes \sigma_x \otimes \sigma_y \ldots$

(J) is **decidable** within (H_p) **iff** measurement outcomes are **definite**
(measurement operator commute with all the operators defining the initial state)

The 2-bit axiom

Axiom: $f_1(0) = f_2(0) \quad f_1(1) = f_2(1)$

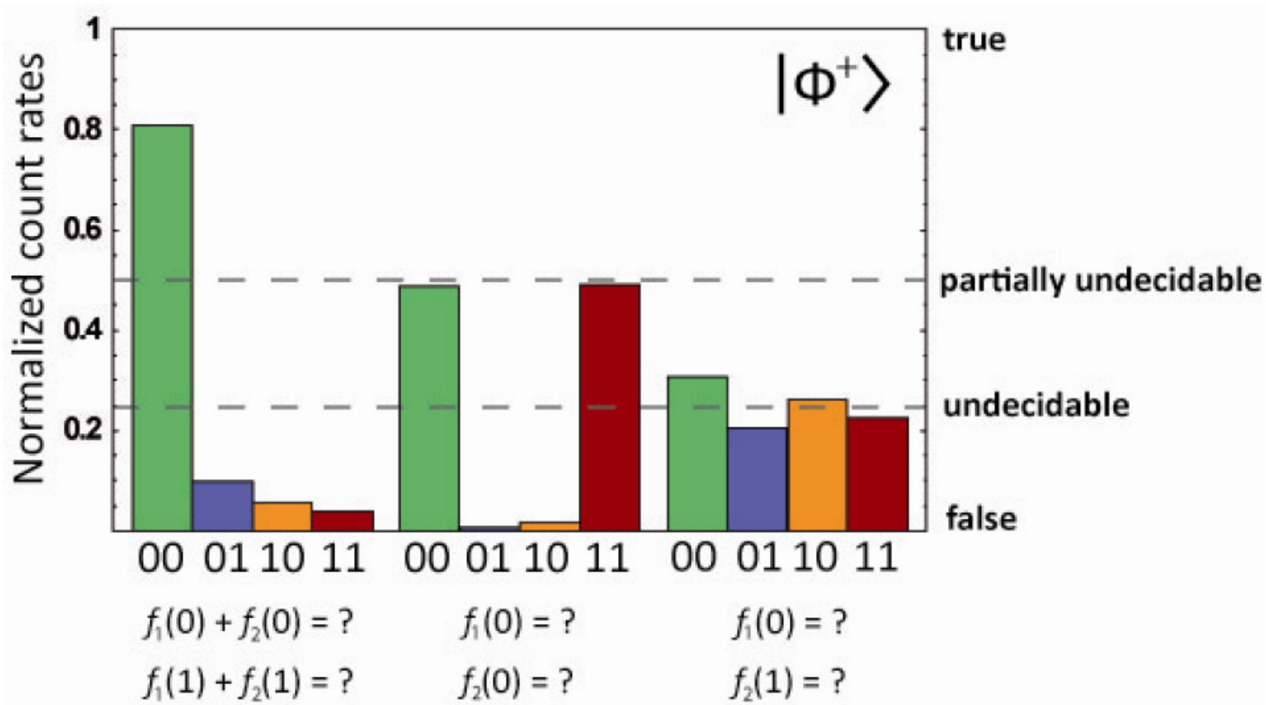
global properties, requires entangled states

Undecidable: $f_1(0) = 0 \quad f_2(1) = 0$

local properties

„Partially" Undecidable: $f_1(0) = 0 \quad f_2(0) = 0$

Full and partial undecidability



Quantum randomness from mathematical undecidability

1. When measured, quantum systems inevitable give results, e.g. „clicks“ in the detectors

2. The clicks must not contain any information whatsoever about the truth value of the undecidable proposition.

They must be random

What is the truth value?

$$|GHZ\rangle = (|z+\rangle_1|z+\rangle_2|z+\rangle_3 + |z-\rangle_1|z-\rangle_2|z-\rangle_3)/\sqrt{2}$$

$$f_1(0)+f_1(1)+f_2(0)+f_2(1)+f_3(1)=1 \qquad \sigma_y \otimes \sigma_y \otimes \sigma_x$$

$$f_1(0)+f_1(1)+f_2(1)+f_3(0)+f_3(1)=1 \qquad \sigma_y \otimes \sigma_x \otimes \sigma_y$$

$$f_1(1)+f_2(0)+f_2(1)+f_3(0)+f_3(1)=1 \qquad \sigma_x \otimes \sigma_y \otimes \sigma_y$$

Logic: $f_1(1)+f_2(1)+f_3(1)=1$

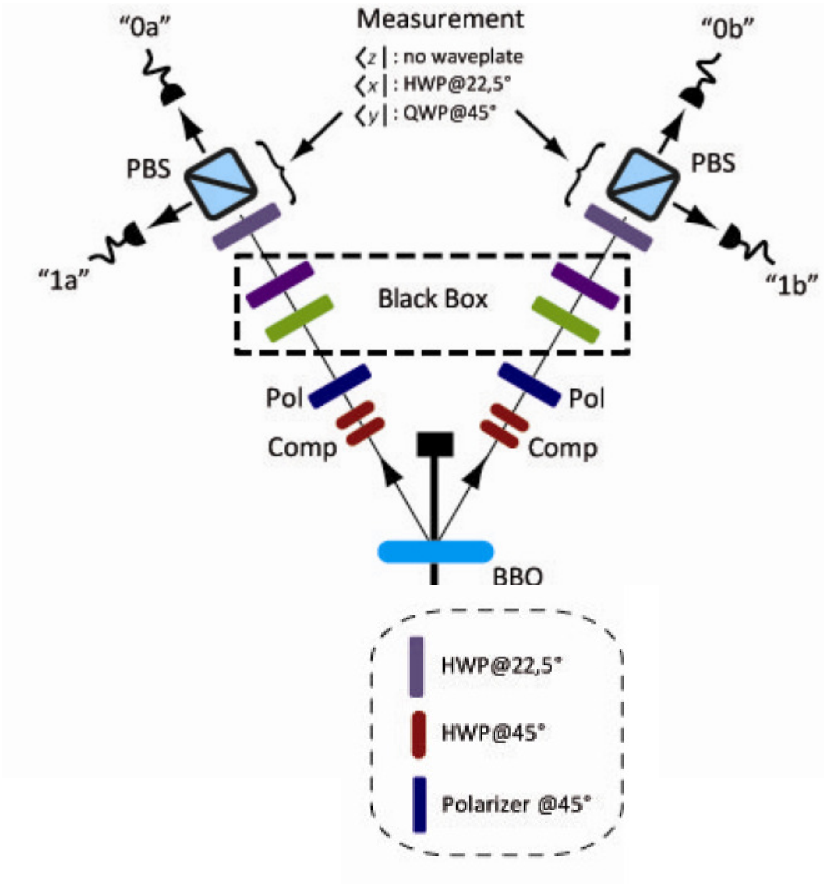
Quantum: $f_1(1)+f_2(1)+f_3(1)=0 \qquad \sigma_x \otimes \sigma_x \otimes \sigma_x$!

Decidable propositions may have „wrong“ values!

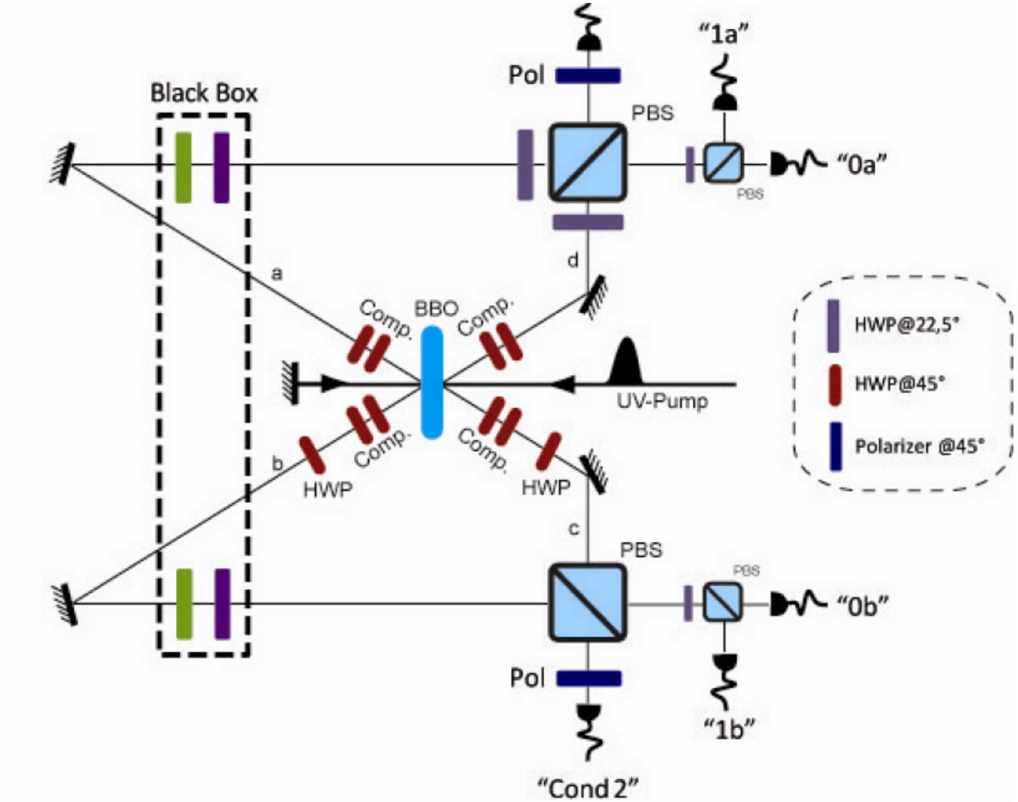
Conclusions & Future

- Quantum experiments can be used to test (un)decidability of mathematical propositions
- Decidable propositions may have “wrong” truth values
- Quantum randomness is a physical manifestation of mathematical undecidability (in Pauli product measurements)
- What about other measurements?

Set-ups



"0a"
Measurement
⟨z| : no waveplate
⟨x| : HWP@22,5°
⟨y| : QWP@45°
"0b"
PBS
PBS
"1a"
"1b"
Black Box
Pol
Pol
Comp
Comp
BBO
HWP@22,5°
HWP@45°
Polarizer @45°

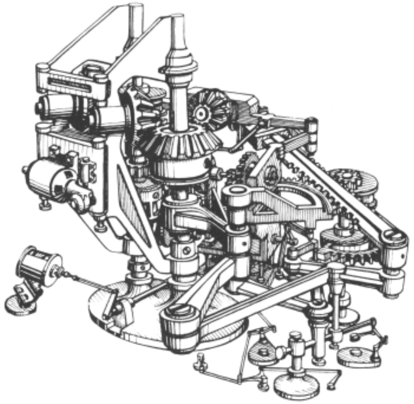


Black Box
Pol
PBS
"1a"
"0a"
PBS
a
d
BBO
Comp.
Comp.
Comp.
Comp.
UV-Pump
b
HWP
HWP
c
PBS
"0b"
PBS
Pol
"1b"
"Cond 2"
HWP@22,5°
HWP@45°
Polarizer @45°

Realism & Locality

Realism: the measurement results are determined by “hidden variables” which exist prior to and independent of observation.

„World“



„Passive“ Observer

Locality: the results obtained at one location are independent of any measurements or actions performed at space-like separated regions.

Testing the truth-value

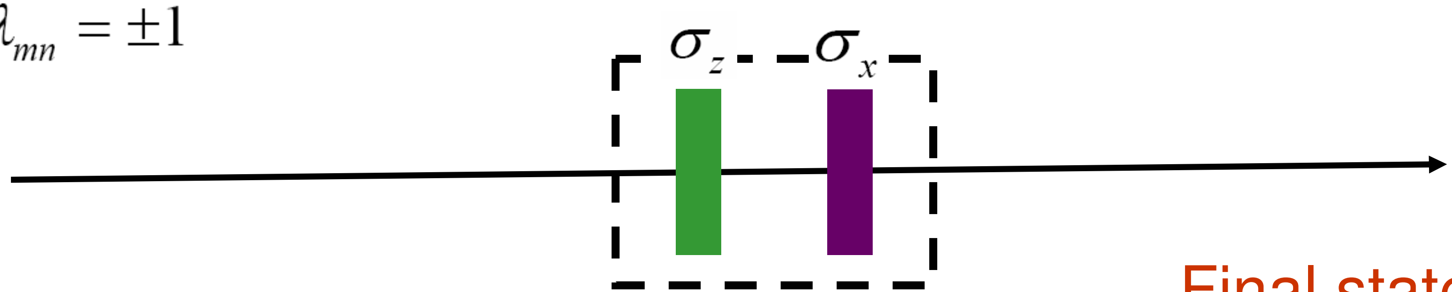
Initial state

$$\rho = \tfrac{1}{2}[\mathbf{1} + \lambda_{mn} i^{mn} \sigma_x^m \sigma_z^n]$$

$$\lambda_{mn} = \pm 1$$

Black box action

$$U = \sigma_x^{f(0)} \sigma_z^{f(1)}$$



Final state

$$\rho \;\rightarrow\; U \rho U^{+} = \tfrac{1}{2}[\mathbf{1} + \lambda_{mn} (-1)^{n f(0) + m f(1)} i^{mn} \sigma_x^m \sigma_z^n]$$

Checking the truth value of the proposition:

“The value of $n\, f(0) + m\, f(1) = 0$“